Examples of L'Hopital's rule

1)

$$\lim_{x \to 0} \frac{\sin x}{x}$$

$$\stackrel{LH}{=} \lim_{x \to 0} \frac{\cos x}{1}$$

$$= \cos 0 = 1$$

2) $$\lim_{x \to \infty} \frac{\ln x}{x}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1}$$

$$= \frac{1}{\infty} = 0.$$

Ex 1 $\quad X \sim \text{Bernoulli}(p)$.

$$X = \begin{cases} 1 & \text{w.p} \quad p \\ 0 & \text{w.p} \quad 1-p \end{cases}$$

Does the ETT hold?

$$w(F) = 1$$

$$\lim_{k \to 1} \frac{P(X=1)}{1 - F(1-1)}$$

$$= \cancel{\lim} \frac{P(X=1)}{1 - F(0)}$$

$$= \frac{p}{1 - (1-p)}$$

$$= \frac{p}{p}$$

$$= 1 \neq 0$$

ETT fails to hold.

Ex 2

Suppose X has PMF

$$P(X=k) = \begin{cases} 1 & \text{if } k = k_0 \\ 0 & \text{if } k \neq k_0 \end{cases}$$

Does the ETT hold?

$\cancel{w(F) = 1}$ $\quad w(F) = k_0$

$$\lim_{k \to k_0} \frac{P(X=k)}{1 - F(k-1)}$$

$$= \frac{P(X=k_0)}{1 - F(k_0 - 1)}$$

$$= \frac{1}{1 - 0}$$

$$= 1 \neq 0$$

$\Rightarrow$ ETT fails to hold.

Ex 3

Suppose $X \sim \text{Poisson}(\lambda)$.

Does the ETT hold?

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, \ldots$$

$$\omega(F) = +\infty$$

$$\lim_{k \to \infty} \frac{P(X=k)}{1 - F(k-1)}$$

$$= \lim_{k \to \infty} \frac{P(X=k)}{1 - P(X \leq k-1)}$$

$$= \lim_{k \to \infty} \frac{P(X=k)}{P(X \geq k)}$$

$$= \lim_{k \to \infty} \frac{P(X=k)}{\sum_{j=k}^{\infty} P(X=j)}$$

$$= \lim_{k \to \infty} \frac{\frac{\cancel{e^{-\lambda}} \lambda^k}{k!}}{\sum_{j=k}^{\infty} \frac{\cancel{e^{-\lambda}} \lambda^j}{j!}}$$

$$= \lim_{k \to \infty} \frac{\frac{\lambda^k}{k!}}{\sum_{j=k}^{\infty} \frac{\lambda^j}{j!}}$$

$$= \lim_{k \to \infty} \frac{1}{\sum_{j=k}^{\infty} \frac{k!\, \lambda^{j-k}}{j!}}$$

$$= \lim_{k \to \infty} \frac{1}{\sum_{j=k}^{\infty} \frac{\cancel{1 \cdots k}}{\cancel{1 \cdots k} \cdots j} \lambda^{j-k}}$$

$$= \lim_{k \to \infty} \frac{1}{\sum_{j=k}^{\infty} \frac{1}{(k+1) \cdots j} \lambda^{j-k}}$$

$$= \lim_{k \to \infty} \frac{1}{\sum_{j=k}^{\infty} \frac{1}{\underset{k}{\underset{\vee\!/}{\boxed{(k+1)}}} \cdots \underset{k}{\underset{\vee\!/}{\boxed{(k+j-k)}}}} \lambda^{j-k}}$$

$$\geqslant \lim_{k \to \infty} \frac{1}{\sum_{j=k}^{\infty} \frac{1}{k^{j-k}} \lambda^{j-k}}$$

$\boxed{\text{set } m = j-k}$

$$= \lim_{k \to \infty} \frac{1}{\sum_{m=0}^{\infty} \left(\frac{\lambda}{k}\right)^{m}}$$

$$\boxed{\sum_{m=0}^{\infty} a^m = \frac{1}{1-a}}$$

$$= \lim_{k \to \infty} \frac{1}{\frac{1}{1-\frac{\lambda}{k}}}$$

$$= \frac{1}{\frac{1}{1-0}}$$

$$= 1$$

Hence ETT fails to holds.