Example 1

$$F(x) = \Phi(x), \quad -\infty < x < \infty$$

CDF of $N(0,1)$

$$w(F) = ?$$

$$F(x) = 1$$

$$\Rightarrow \Phi(x) = 1$$

$$\Rightarrow \quad x = \Phi^{-1}(1)$$

$$= +\infty$$

$$I: \lim_{t \to w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{1 - \Phi(t + x\gamma(t))}{1 - \Phi(t)}$$

$$\overset{LH}{\underset{\text{rule}}{=}} \lim_{t \to \infty} \frac{-\phi(t + x\gamma(t))\,(1 + x\gamma'(t))}{-\phi(t)}$$

$$\boxed{\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}}$$

$$= \lim_{t \to \infty} \frac{\cancel{\frac{1}{\sqrt{2\pi}}}\, e^{-\frac{(t + x\gamma(t))^2}{2}}\,(1 + x\gamma'(t))}{\cancel{\frac{1}{\sqrt{2\pi}}}\, e^{-\frac{t^2}{2}}}$$

$$= \lim_{t \to \infty} e^{\frac{t^2}{2} - \frac{(t + x\gamma(t))^2}{2}} (1 + x\gamma'(t))$$

$$= \lim_{t \to \infty} e^{\frac{\cancel{t^2} - \cancel{t^2} - 2x\gamma(t)t - x^2(\gamma(t))^2}{2}} (1 + x\gamma'(t))$$

$$= \lim_{t \to \infty} e^{-x\gamma(t)t - \frac{x^2(\gamma(t))^2}{2}} (1 + x\gamma'(t))$$

$$\boxed{\begin{aligned} \gamma(t) &= \frac{1}{t} \\ \gamma'(t) &= -\frac{1}{t^2} \end{aligned}}$$

$$= \lim_{t \to \infty} e^{-x - \frac{x^2}{2t^2}} \left(1 + x\left(-\frac{1}{t^2}\right)\right)$$

$$\to 0 \qquad \leftarrow$$

$$= e^{-x}$$

Hence, the Gumbel type is attained. That is, there exists $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$[F(a_n x + b_n)]^n \longrightarrow e^{-e^{-x}}$$

as $n \to \infty$

$$a_n = \gamma\left(F^{-1}\left(1-\frac{1}{n}\right)\right)$$

$$b_n = F^{-1}\left(1-\frac{1}{n}\right)$$

Set $F(x) = 1-\frac{1}{n}$

$$\Rightarrow \Phi(x) = 1-\frac{1}{n}$$

$$\Rightarrow x = \Phi^{-1}\left(1-\frac{1}{n}\right)$$

$$\Rightarrow F^{-1}\left(1-\frac{1}{n}\right) = \Phi^{-1}\left(1-\frac{1}{n}\right).$$

$$\Rightarrow a_n = \frac{1}{F^{-1}\left(1-\frac{1}{n}\right)} = \frac{1}{\Phi^{-1}\left(1-\frac{1}{n}\right)}$$

and $b_n = \Phi^{-1}\left(1-\frac{1}{n}\right)$.