Quantitative Finance

ISSN: 1469-7688 (Print) 1469-7696 (Online) Journal homepage: http://www.tandfonline.com/loi/rquf20

Modelling exchange rate returns: which flexible distribution to use?

Canan G. Corlu & Alper Corlu

Quantitative Finance, 2015
Vol. 15, No. 11, 1851–1864, http://dx.doi.org/10.1080/14697688.2014.942231

Routledge
Taylor & Francis Group

Modelling exchange rate returns: which flexible distribution to use?

CANAN G. CORLU*† and ALPER CORLU‡

†Metropolitan College, Boston University, Boston, MA 02109, USA
‡RBS Citizens, Boston, MA 02109, USA

(*Received 11 September 2013; accepted 1 July 2014*)

It is well known that the normal distribution is inadequate in capturing the skewed and heavy-tailed behaviour of exchange rate returns. To this end, various flexible distributions that are capable of modelling the asymmetric and tailed behaviour of returns have been proposed. In this paper, we investigate the performance of the generalized lambda distribution (GLD) to capture the skewed and leptokurtic behaviour of exchange rate returns. We do this by conducting a comprehensive numerical study to compare the performance of the GLD against the performances of the skewed t distribution, the unbounded Johnson family of distributions and the normal inverse Gaussian (NIG) distribution. Our results suggest that in terms of the value-at-risk and expected shortfall, the GLD shows at least similar performance to the skewed t distribution and the NIG distribution. Considering the ease in GLD's use for random variate generation in Monte Carlo simulations, we conclude that the GLD can be a good alternative in various financial applications where modelling of the heavy tail behaviour is critical.

Keywords: Exchange rate returns; Generalized lambda distribution; Skewed t distribution; Johnson family of distributions; Normal inverse Gaussian distribution; Risk management

1. Introduction

It is well-known that the normal distribution is inadequate to model the fat tailed and often skewed behaviour of financial returns. Early empirical evidence about this non-normal behaviour (i.e. fat tails and greater kurtosis compared to the normal distribution) can be found in Mandelbrot (1963), Fama (1965), Farber *et al*. (1977), Westerfield (1977), and McFarland *et al*. (1982) among others.

In the case of exchange rates, the fat tails have been studied extensively. Boothe and Glassman (1987) compare the fits of the symmetric stable Paretian, the Student, a mixture of two normals with differing variances, and the normal distribution for modelling four exchange rates versus the US dollar, and find evidence for favouring the Student and the mixture of two normals over the others. Tucker and Pond (1988) investigate four distributions including the scaled Student t distribution, the general stable distribution, the compound normal distribution and the mixed jump diffusion model for characterizing daily changes of six major currencies. Their results show that the mixed jump diffusion model outperforms others. In a related study, Akgiray and Booth (1988) also favour the mixed jump diffusion process over the stable distributions and the mixtures of normals.

In the context of measuring value-at-risk (VaR), Hull and White (1998) use the exponentially weighted moving average model based on a mixture of two normals to characterize the behaviour of 12 exchange rates. Johnson and Scott (1999) compare the performance of the mixed jump diffusion model, discrete mixtures of normal distributions and four alternative versions of the GARCH model in fitting the daily returns of four major currencies. Their results support the use of the mixed jump diffusion model and the mixture of two normals. Huisman *et al*. (2002) find that the Student t distribution is an accurate approximation to the tail behaviour of the exchange rate returns. More recently, Gurrola (2008) compares the fit of the unbounded Johnson family to the mixtures of two normals and the skewed Student t in modelling six major trading currencies and two Latin American currencies. The results favour the skewed Student t in general, while the unbounded Johnson family outperforms others in terms of the VaR performance.

The superior performance of the unbounded Johnson family in estimating VaR has also been shown in Choi (2001) and Simonato (2011). Choi and Nam (2008) use unbounded

*Corresponding author. Email: canan@bu.edu
The opinions expressed in this work are solely those of the two authors and do not represent in any way those of their current and past employers.

Johnson family to estimate the univariate and multivariate GARCH models in modelling several series of daily exchange rates. The authors present evidence that the unbounded Johnson family outperforms the normal and Student *t* distribution in characterizing the tail behaviour of daily exchange rates.

Despite not being exhaustive, the review of the previous research shows that the Student *t* distribution and the unbounded Johnson family (S_U) are the most favoured two distributions in modelling daily exchange rates. However, there are other flexible distributions in the finance literature that have the ability to capture a wide variety of distributional shapes and thus have been successful in modelling financial returns (not necessarily exchange rate returns). Example distributions are the generalized lambda distribution (GLD) and the normal inverse Gaussian (NIG) distribution.

The NIG distribution has a relatively long history in the finance literature starting with Eberlein and Keller (1995). Among others, Prause (1999) shows the applicability of the NIG in modelling German stock and US Stock Index data. Bolviken and Benth (2000) use the NIG to model eight Norwegian stocks. Lillestol (2000) uses the NIG in the context of risk analysis and portfolio choice. Barndorff-Nielsen and Prause (2001) show the ability of the NIG in fitting stock returns and the US/DEM exchange rate.

The GLD has also been used in different contexts in the finance literature. Corrado (2001) uses it in the context of option pricing, while Tarsitano (2004) uses the GLD for modelling income data. More recently, Chalabi *et al*. (2010) propose the GLD as an alternative to the stable distribution and the Student *t* distribution in modelling equities from NASDAQ -100 index. Lee (2003) shows that the GLD is a good candidate for modelling spot exchange rates. However, to the best of our knowledge, its performance in fitting daily exchange rate returns has not been compared to other flexible distributions. Our primary goal in this paper is to investigate the ability of the GLD in capturing the leptokurtic and skewed behaviour of exchange rate returns as an alternative to the more widely used S_U, (skewed) Student *t* distribution, and NIG. To this end, we perform a comprehensive numerical study to model the daily exchange rates of nine currencies for the period 2006–2011. The fits of the distributions are compared using Kolmogorov-Smirnov (KS) test statistic, Anderson-Darling (AD) test statistic and visual plots. Additionally, the performance of the models in risk estimation is investigated using in-sample VaR failure rates and ES as risk measures.

Various forms of the skewed Student *t* distribution have also been used extensively in finance. Hansen (1994) is the first to propose a skewed *t* distribution for modelling returns. This study is followed by Fernandez and Steel (1998), Branco and Dey (2001), Venter and de Jongh (2002), Azzalini and Capitanio (2003), Jones and Faddy (2003), Sahu *et al*. (2003), Patton (2004), and Bauwens and Laurent (2005). The skewed Student *t* distribution (Skewed *t*) proposed in Azzalini and Capitanio (2003) is used in this paper.

We organize the remainder of the paper as follows. Section 2 presents the flexible distributions of interest. Section 3 discusses the exchange rate return data and the fitting methods used to fit the data. Section 4 presents our experimental setting and findings. Section 5 concludes.

2. Flexible distributions

We describe the GLD in section 2.1, the Johnson translation system in section 2.2, the Skewed *t* in section 2.3 and the NIG in section 2.4.

2.1. The generalized lambda distribution

The GLD (Joiner and Rosenblatt 1971, Ramberg and Schmeiser 1974, Filliben 1975), which is an extension of Tukey's lambda distribution (Hastings *et al*. 1947) is defined by the following inverse cumulative distribution function (cdf):

$$F^{-1}(u; \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2} \quad (1)$$

where $0 \le u \le 1$; λ_1 is the location parameter, λ_2 is the scale parameter, λ_3 and λ_4 are related to skewness and kurtosis, respectively. This representation is denoted Ramberg-Schmeiser Generalized Lambda Distribution (RS GLD) referring to the parameterization of Ramberg and Schmeiser (1974). However, the probability density function (pdf) related to (1) does not specify a proper pdf for all combinations of the shape parameters λ_3 and λ_4 (see Fournier *et al*. (2007) for the six regions in which a proper pdf is well defined). This limitation of the RS GLD becomes problematic especially when estimating the parameters of the GLD. More specifically, any (λ_3, λ_4) estimate combination that is not part of the specified six regions would not produce a valid pdf. In order to avoid this problem, Freimer *et al*. (1988) propose a different parameterization for the GLD denoted Freimer-Mudholkar-Kollia-Lin Generalized Lambda Distribution (FMKL GLD), which is given by

$$F^{-1}(u; \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3} - 1}{\lambda_3} - \frac{(1-u)^{\lambda_4} - 1}{\lambda_4}\right) \quad (2)$$

This parameterization is well defined for all parameter values; the only restriction is that $\lambda_2 > 0$. Also, in order to have a finite kth moment, the additional requirement is $min\ (\lambda_3, \lambda_4) > -1/k$. The definitions of the FMKL GLD parameters are similar to those of the RS GLD. Both of these representations can present a wide variety of shapes and therefore are utilized in practice; however, generally the FMKL GLD is preferred due to the ease in its use. In this paper, we also use the FMKL GLD representation.

The FMKL GLD curves can be categorized into five categories depending on the variety of shapes that can be represented by several combinations of the shape parameters λ_3 and λ_4 (Freimer *et al*. 1988). In particular, Class-I family ($\lambda_3 < 1, \lambda_4 < 1$) represents unimodal densities with continuous tails, Class-II family ($\lambda_3 > 1, \lambda_4 < 1$) represents monotone pdfs similar to exponential distribution, Class-III family ($1 < \lambda_3 < 2, 1 < \lambda_4 < 2$) represents U-shaped densities with truncated tails, Class-IV family ($\lambda_3 > 2, 1 < \lambda_4 < 2$) represents S-shaped densities and Class-V family ($\lambda_3 > 2, \lambda_4 > 2$) represents unimodal densities with truncated tails. We find that the exchange rate return data belongs to Class-I family (see section 3).

Since GLD is represented with a quantile function, it is straightforward to obtain a closed-form expression for VaR and ES:

$$\begin{aligned}\mathrm{VaR}(\alpha) &= F^{-1}(\alpha;\lambda_1,\lambda_2,\lambda_3,\lambda_4)\\ &= \lambda_1 + \frac{1}{\lambda_2}\left(\frac{\alpha^{\lambda_3}-1}{\lambda_3} - \frac{(1-\alpha)^{\lambda_4}-1}{\lambda_4}\right) \quad (3)\\ \mathrm{ES}(\alpha) &= \frac{1}{\alpha}\int_0^\alpha F^{-1}(y;\lambda_1,\lambda_2,\lambda_3,\lambda_4)dy\\ &= \lambda_1 + \frac{1}{\alpha\lambda_2\lambda_3}\left(\frac{\alpha^{\lambda_3+1}}{\lambda_3+1}-\alpha\right)\\ &\quad - \frac{1}{\alpha\lambda_2\lambda_4}\left(\frac{1-(1-\alpha)^{\lambda_4+1}}{\lambda_4+1}-\alpha\right) \quad (4)\end{aligned}$$

2.2. *The Johnson translation system*

A random variable X from the Johnson translation system is represented by (Johnson 1949)

$$X = \xi + \lambda r^{-1}\left(\frac{Z-\gamma}{\delta}\right)$$

where Z is a standard normal random variable, γ and δ are shape parameters, ξ is a location parameter, λ is a scale parameter and $r(\cdot)$ is one of the following transformations:

$$r(y) = \begin{cases} y & \text{for the } S_N(\text{normal}) \text{ family,}\\ \log(y) & \text{for the } S_L(\text{lognormal}) \text{ family,}\\ \log(y/(1-y)) & \text{for the } S_B(\text{bounded}) \text{ family,}\\ \log(y+\sqrt{y^2+1}) & \text{for the } S_U(\text{unbounded}) \text{ family}\end{cases}$$

The range of the random variable X is defined by the family of interest: $X > \xi$ and $\lambda = 1$ for the S_L family; $\xi < X < \xi + \lambda$ for the S_B family; and $-\infty < X < \infty$ for the S_N and S_U families. There is a unique family; i.e. choice of r, for each feasible combination of the skewness and kurtosis values. The exchange rate returns considered in this paper have skewness and kurtosis values that correspond to the S_U family. Thus, we only consider the S_U family of the Johnson translation system.

The cdf of a Johnson random variable X is given by

$$F(X) = \Phi\left(\gamma + \delta r\left(\frac{X-\xi}{\lambda}\right)\right)$$

where Φ stands for the standard normal distribution function. Thus, VaR(α) is equal to

$$\xi + \lambda r^{-1}\left(\frac{\Phi^{-1}(\alpha)-\gamma}{\delta}\right)$$

For S_U distribution, since $r^{-1}(y) = (e^y - e^{-y})/2$, VaR($\alpha$) is given by

$$\xi + \lambda\left(\frac{\exp\left(\frac{\Phi^{-1}(\alpha)-\gamma}{\delta}\right) - \exp\left(\frac{\gamma-\Phi^{-1}(\alpha)}{\delta}\right)}{2}\right)$$

2.3. *The skewed student t distribution*

A number of skewed Student *t* distributions have been proposed in the literature. We follow the parameterization in Azzalini and Capitanio (2003) due to its simplicity in implementation. A random variable X from the skewed Student *t* distribution has a density of the form

$$\begin{aligned}&f(x;\delta,\nu,\mu,\beta)\\ &= \frac{1}{\delta}t_\nu\left(\frac{x-\mu}{\delta}\right)2T_{\nu+1}\left(\beta\left(\frac{x-\mu}{\delta}\right)\sqrt{\frac{\nu+1}{\left(\frac{x-\mu}{\delta}\right)^2+\nu}}\right)\end{aligned}$$

where t_ν is the density of standard Student *t* distribution with ν degrees of freedom and $T_{\nu+1}$ is the distribution function of the standard Student *t* distribution with $\nu+1$ degrees of freedom.

2.4. *The NIG distribution*

The NIG is an extension of a more general distribution called generalized hyperbolic distribution (GHD) whose density is given by

$$\begin{aligned}&f(x;\lambda,\alpha,\beta,\mu,\delta)\\ &= \frac{(\delta\sqrt{\alpha^2-\beta^2})^\lambda(\delta\alpha)^{1/2-\lambda}}{\sqrt{2\pi}\delta K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}\left(1+\frac{(x-\mu)^2}{\delta^2}\right)^{\lambda/2-1/4}\\ &\quad\times\exp(\beta(x-\mu))K_{\lambda-1/2}\left(\alpha\delta\sqrt{1+\frac{(x-\mu)^2}{\delta^2}}\right)\end{aligned}$$

where K_λ is the modified third-order Bessel function. The density is defined under the following parameter restrictions:

$$\begin{aligned}&\delta \ge 0 \text{ and } |\beta| < \alpha \text{ if } \lambda > 0\\ &\delta > 0 \text{ and } |\beta| < \alpha \text{ if } \lambda = 0\\ &\delta > 0 \text{ and } |\beta| \le \alpha \text{ if } \lambda < 0\end{aligned}$$

It has been shown in Pfaff (2012) that GHD not only represents semi-heavy tails but also skewed distributions. The variants of the GHD can be obtained by the changing values of the parameter λ; that is why, λ is called the class-defining parameter.

The NIG distribution can be obtained from the GHD by setting $\lambda = -1/2$ and its density is given by

$$\begin{aligned}&f(x;\alpha,\beta,\mu,\delta)\\ &= \frac{\alpha\delta K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\pi\sqrt{\delta^2+(x-\mu)^2}}e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)}\end{aligned}$$

with $|\beta| \le \alpha$ and $\delta > 0$ (Prause 1999).

3. Exchange rate return data and fitting procedures

Our data consists of nine different currency exchange rates in terms of the US dollar. These currencies include the Australian dollar (AUD), the Brazilian real (BRL), the Canadian dollar (CAD), the Swiss franc (CHF), the euro (EUR), the sterling (GBP), the Mexican peso (MXN), the Turkish lira (TRY) and the Japanese Yen (JPY). All data-sets are daily closing rates from DataStream covering a six-year period from 2 January 2006 to 30 December 2011, which corresponds to a total of 1565 observations.

We transform the data-set into daily logarithmic returns, $X_t = \log(S_t/S_{t-1})$ where S_t is the level of the daily exchange rate at time t. Table 1 presents preliminary statistics on the data.

Table 1. Statistical properties of the log-return series.

Currency	Mean	Standard deviation	Skewness	Excess Kurtosis	Bera-Jarque
AUD	0.0002	0.0105	−1.0058	11.4628	8832.00
BRL	0.0001	0.0102	−0.5848	13.6094	12167.01
CAD	0.0001	0.0070	−0.1445	3.1836	666.36
CHF	0.0002	0.0075	−0.7379	13.7407	12453.87
EUR	0.0001	0.0068	0.1516	3.6169	859.09
GBP	−0.0001	0.0069	−0.0575	4.3617	1241.45
JPY	0.0002	0.0071	0.3545	4.3946	1292.13
MXN	−0.0001	0.0076	−0.7936	13.4877	12026.93
TRY	−0.0002	0.0095	−0.5571	7.4040	3655.66

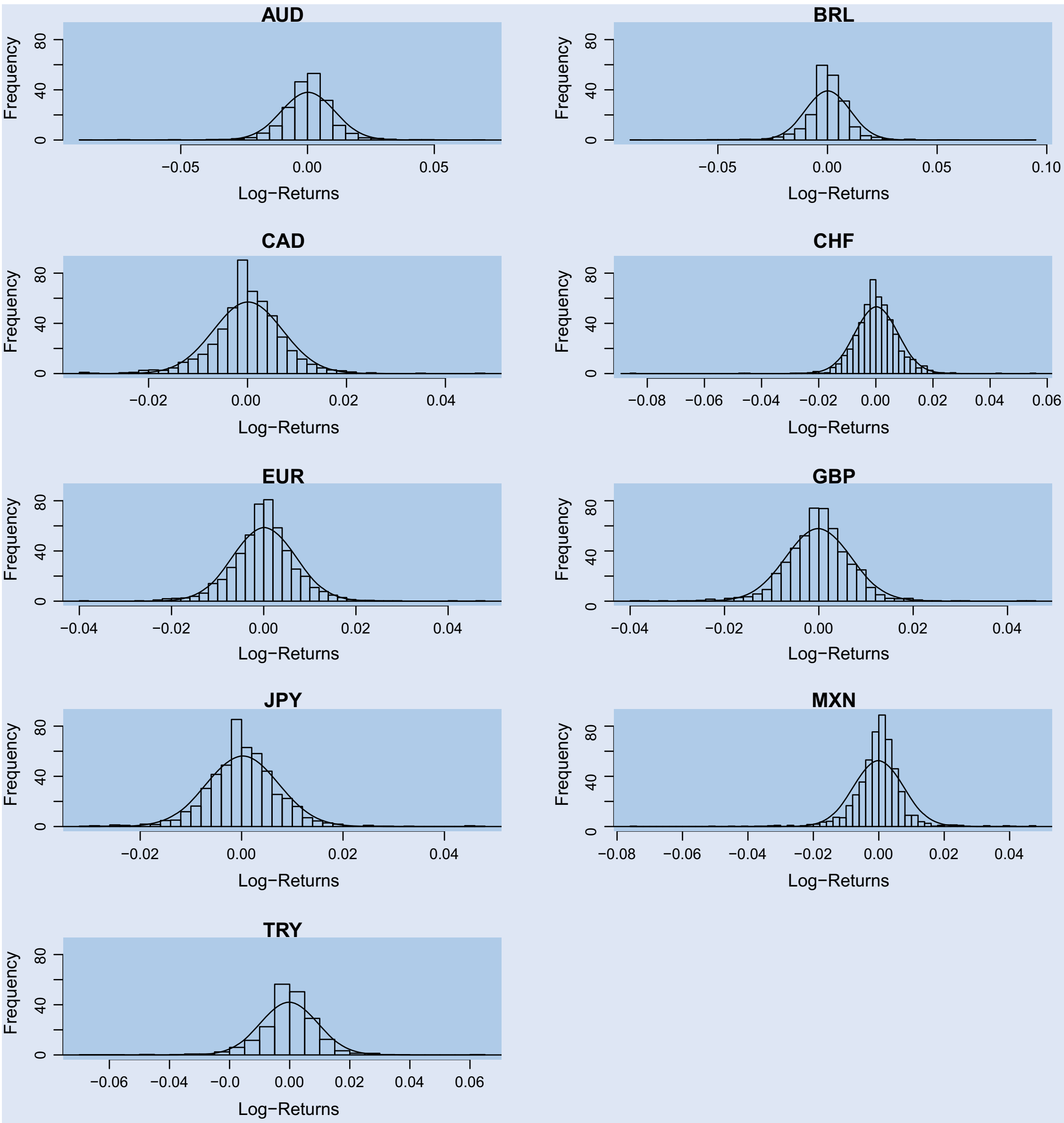

Figure 1. Observed frequencies versus the normal distribution density.

The second and the third column of table 1 present the mean and the standard deviation of the log-returns, while the fourth and the fifth columns give the skewness and the excess kurtosis, respectively. The skewness equals to $s = m_3/m_2^{3/2}$ and the excess kurtosis is given by $k = (m_4/m_2^2) - 3$, where m_i for $i = 2, 3, 4$ is the estimate of the ith moment around the mean. Assuming normality of log-returns, the standard errors of the skewness and kurtosis estimates are $\sqrt{6/n} = 0.0619$ and $\sqrt{24/n} = 0.124$, respectively, where n is the sample size.

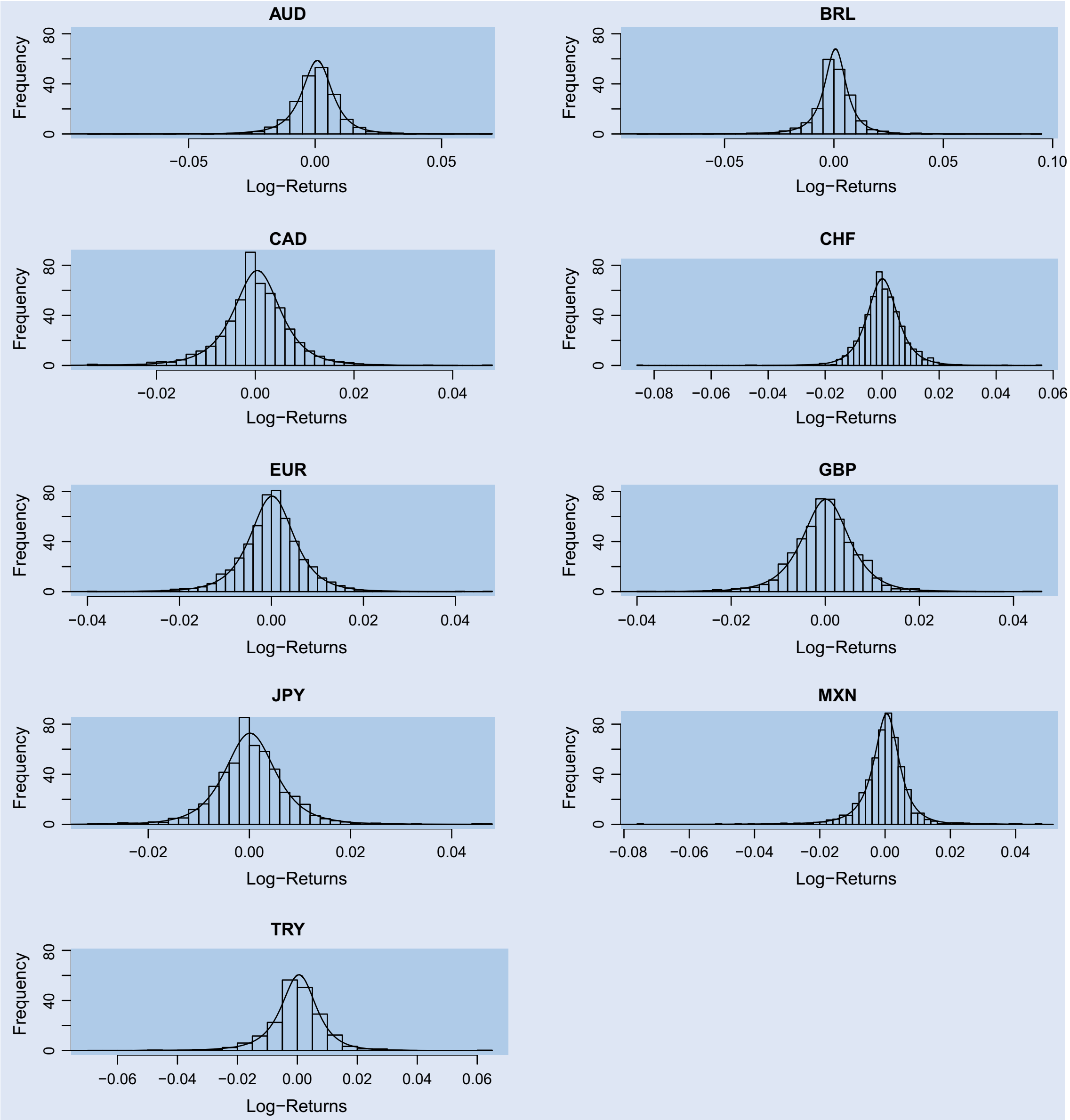

Figure 2. Historical log-return frequencies (bars) and the fitted GLD density (solid lines) for each of the nine currencies.

The last column of table 1 tabulates the Bera-Jarque (BJ) statistic, which is a statistic used to test any departure from normality. The BJ test statistic is calculated by $n(s^2/6+k^2/24)$ and under the null hypothesis of normality, it is asymptotically χ^2 distributed with two degrees of freedom. Under 1% level of significance, the BJ test statistic is 9.21, indicating that all of the nine exchange rate currencies are non-normal. Figure 1, which illustrates the observed frequencies of the currencies along with the normal distribution density obtained using the corresponding mean and standard deviation values of the log-returns (represented with the solid line), presents another evidence to the nonnormal behaviour of the log-returns.

In the remainder of this section, we present the parameter estimation algorithms used to fit each probability distribution of interest to the log-returns. There are a variety of parameter estimation techniques in the literature to estimate the parameters of the distributions. For fairness, we use the maximum likelihood estimation method when possible. However, the maximum likelihood estimation method is problematic for the S_U; i.e. the parameters ξ and λ may violate the standard regularity conditions (Johnson 1949). Hence, the method proposed by Tuenter (2001) is utilized to estimate S_U parameters.

3.1. *Fitting with the GLD*

Due to its flexibility and ability to model a variety of distributional shapes, several fitting methods have been proposed in the literature to estimate the parameters of the GLD. We refer the reader to Corlu and Meterelliyoz (2014) for a comprehensive review of these methods. Most of these methods have been implemented in the GLDEX package of R. We use the maximum likelihood estimation method in the GLDEX package. The details of this method can be found in Su (2007) and we present the resulting parameter estimates in table 2.

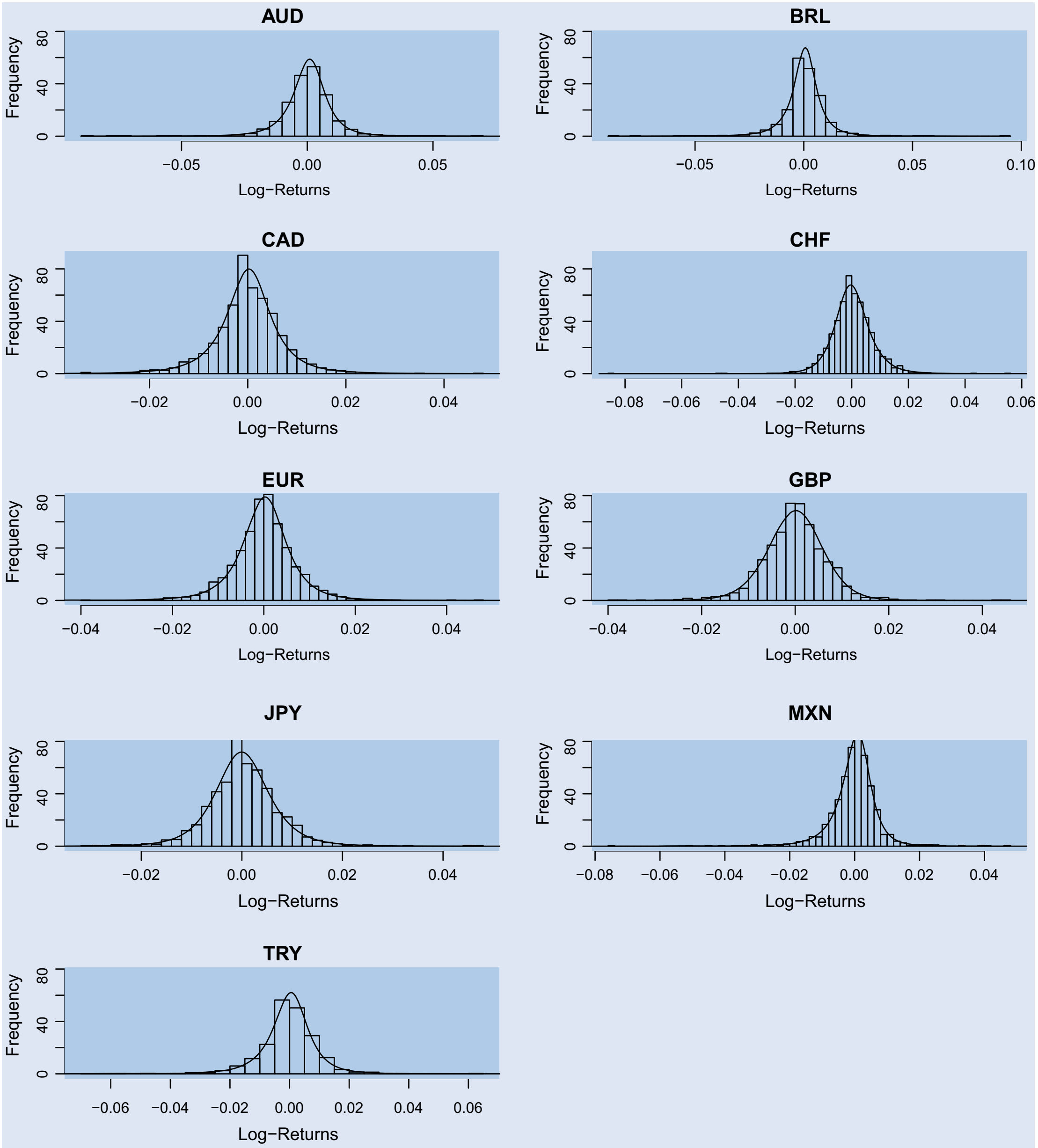

Figure 3. Historical log-return frequencies (bars) and the fitted S_U density (solid lines) for each of the nine currencies.

3.2. Fitting with the unbounded Johnson family

Several fitting methods have been devised for estimating the parameters of the S_U. A good review of these fitting methods can be found in Biller and Corlu (2012). Due to its simplicity yet satisfactory performance compared to other fitting methods, we use the method proposed in Tuenter (2001) in this paper.

Tuenter's algorithm is similar to the method of moments, which estimates the parameters by equalizing the sample skewness and sample kurtosis of the data to the theoretical skewness $\sqrt{\beta_1}$ and theoretical kurtosis β_2 given, respectively, by

$$\sqrt{\beta_1} = \omega(\omega-1)\frac{(\omega(\omega+2)\sinh 3\Omega + 3\sinh\Omega)^2}{2(\omega\cosh 2\Omega+1)^3}$$

$$\beta_2 = \frac{\omega^2(\omega^4+2\omega^3+3\omega^2-3)\cosh 4\Omega + 4\omega^2(\omega+2)\cosh 2\Omega + 3(2\omega+1)}{2(\omega\cosh 2\Omega+1)^2}$$

where $\omega = \exp(\delta^{-2})$ and $\Omega = \gamma/\delta$. However, Tuenter (2001) simplified this problem to a single root finding procedure, which solves for ω from the following equation:

$$\beta_1 = (\omega - 1 - f(\omega))\left(\omega + 2 + \frac{1}{2}f(\omega)\right)^2$$

where

$$f(\omega) = -2 + \sqrt{4 + 2\left(\omega^2 - \frac{\beta_2 - 3}{\omega^2 + 2\omega + 3}\right)}$$

Using the value of ω obtained from this procedure, we identify Ω by

$$\Omega = -\mathrm{sgn}(\sqrt{\beta_1})\sinh^{-1}\sqrt{\frac{\omega+1}{2\omega}\left(\frac{\omega-1}{f(\omega)} - 1\right)}$$

After finding the value of ω and Ω, we obtain δ from the relationship $\omega = \exp(\delta^{-2})$ and γ from the relationship $\Omega = \gamma/\delta$. Once γ and δ are obtained, ξ and λ are identified from $\lambda = \sigma_X/\sigma_Y$ and $\xi = \mathrm{E}(X) - \lambda\mathrm{E}(Y)$, where $Y = (X-\xi)/\lambda$. In this representation, σ is the standard deviation, E is the

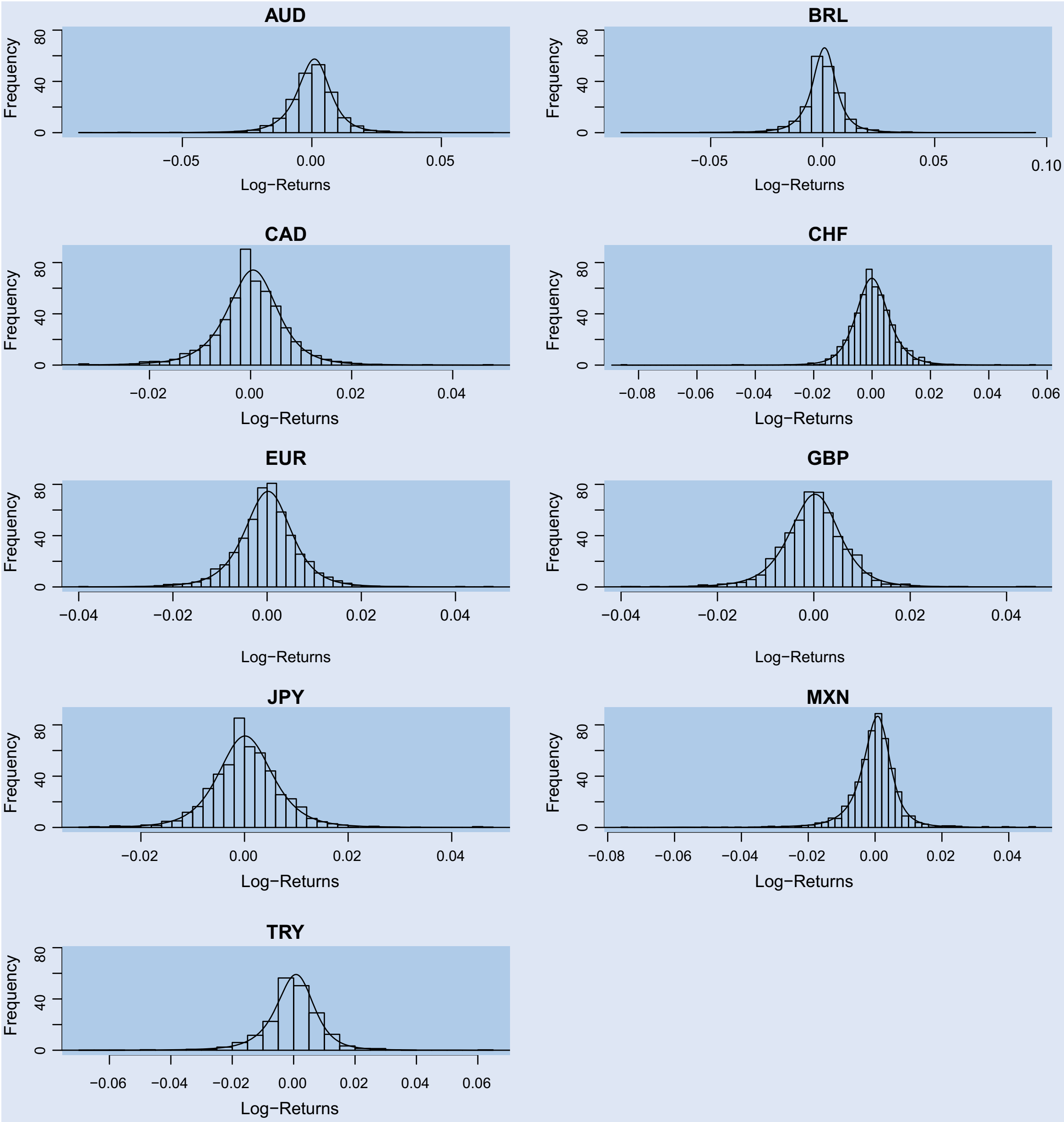

Figure 4. Historical log-return frequencies (bars) and the fitted Skewed *t* density (solid lines) for each of the nine currencies.

expectation operator and X is the Johnson variate. The resulting parameter estimates are given in table 3.

3.3. *Fitting with the skewed student t distribution*

The parameter estimation for the Skewed *t* distribution is performed using the maximum likelihood estimation method described in Azzalini and Capitanio (2003). Assuming the availability of independent and identically distributed (i.i.d.) n observations, $x_1, x_2, \ldots, x_n$, the log-likelihood function is given by

$$- n \log(\delta) + n \log(2) + \sum_{i=1}^{n} \log \left[t_\nu \left(\frac{x_i - \mu}{\delta} \right) \right] + \sum_{i=1}^{n} \log \left[T_{\nu+1} \left(\beta \left(\frac{x_i - \mu}{\delta} \right) \sqrt{\frac{\nu + 1}{\left(\frac{x_i - \mu}{\delta}\right)^2 + \nu}} \right) \right]$$

The maximization of the log-likelihood is accomplished using the Nelder-Mead algorithm. The parameters obtained are tabulated in table 4.

3.4. *Fitting with the NIG distribution*

The typical method for fitting the NIG distribution is the maximum likelihood estimation method. Assuming the availability of i.i.d. historical data of length $n, x_1, x_2, \ldots, x_n$, the log-likelihood function for the NIG distribution is given by

$$n \log(\alpha) + n \log(\delta) + \sum_{i=1}^{n} \log \left[K_1(\alpha + \sqrt{\delta^2 + (x_i - \mu)^2}) \right] - \sum_{i=1}^{n} \log \left[\pi \sqrt{\delta^2 + (x_i - \mu)^2} \right]$$

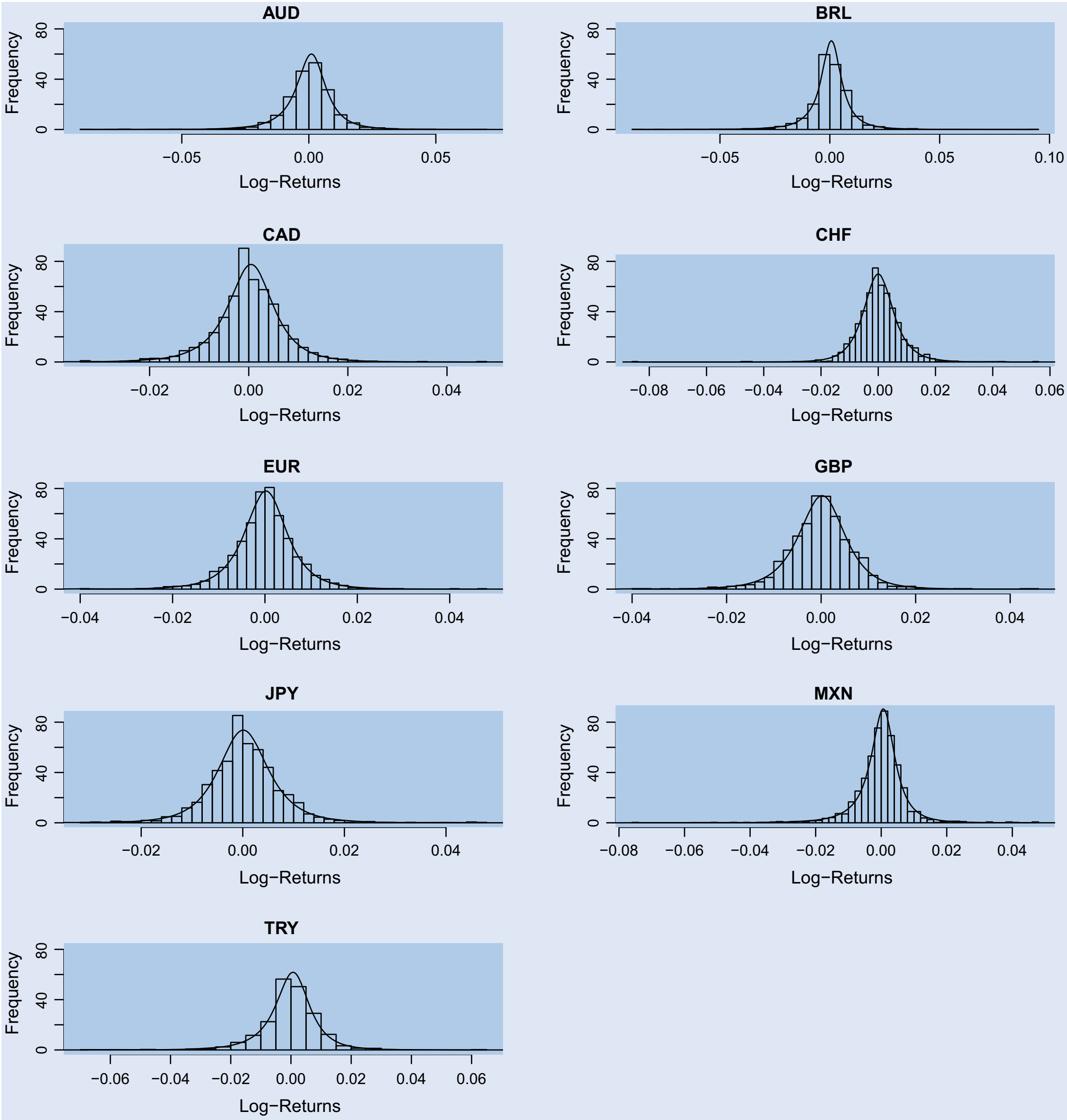

Figure 5. Historical log-return frequencies (bars) and the fitted NIG density (solid lines) for each of the nine currencies.

$$+\sum_{i=1}^{n}\left[\delta\sqrt{\alpha^2-\beta^2}+\beta(x_i-\mu)\right]$$

We use the ghyp package in R, which maximizes the log-likelihoods using the Nelder-Mead algorithm. The resulting parameter estimates are given in table 5.

4. Results and insights

Our goal is to evaluate the performances of the density functions in section 3 to model the log-returns. Section 4.1 presents the KS and AD test statistics as well as visual plots, while section 4.2 presents the VaR and ES levels. In case of GLD fits, the closed-form expressions in (3) and (4) are used for estimating the VaR and ES levels, respectively. For other density functions, VaR and ES levels are computed numerically.

4.1. Goodness-of-fit

We first compare the results using visual plots. Figures 2 –5 present the fitted densities against the observed frequencies of the log-returns for the GLD, S_U, Skewed t, and NIG, respectively. The fitted densities are obtained using the corresponding estimated parameters in tables 2–5 and are drawn with solid lines. We observe that all of the distributions perform very similarly in modelling the data. In general, none of the distributions is able to capture the peakedness of the currencies CAD and JPY, while S_U does a poor job in capturing the peakedness of the GBP.

To calculate the goodness-of-fit, we use the KS test statistic (Chakravant *et al*. 1967) and the AD test statistic (Anderson and Darling 1954). Both of these tests summarize the difference between the fitted cdf $\hat{F}$ and the empirical cdf F_e. In particular, the KS test statistic corresponds to the largest distance between $F_e(x)$ and $\hat{F}(x)$; i.e. $sup_x\{|F_e(x)-\hat{F}(x)|\}$, while the AD test statistic corresponds to the weighted average of the squared

Table 2. Parameter estimates for the GLD.

	λ_1	λ_2	λ_3	λ_4
AUD	0.00070	271.31013	−0.24977	−0.17304
BRL	0.00059	332.99440	−0.33560	−0.25303
CAD	0.00032	333.58870	−0.16745	−0.11042
CHF	0.00011	296.95540	−0.09317	−0.10915
EUR	0.00013	331.03540	−0.12567	−0.11150
GBP	0.00009	316.36630	−0.11830	−0.08380
JPY	0.00017	313.39740	−0.09432	−0.11947
MXN	0.00035	422.99380	−0.32188	−0.20335
TRY	0.00035	274.16910	−0.23281	−0.13051

Table 3. Parameter estimates for the S_U distribution.

	γ	δ	ξ	λ
AUD	0.12740	1.19477	0.00149	0.00808
BRL	0.11007	1.04153	0.00106	0.00613
CAD	0.06036	1.12782	0.00041	0.00563
CHF	−0.21430	1.55136	−0.00125	0.00908
EUR	0.06449	1.27445	0.00047	0.00643
GBP	0.17410	2.54109	0.00103	0.01477
JPY	−0.07226	1.55722	−0.00032	0.00864
MXN	0.32339	1.20202	0.00184	0.00543
TRY	0.15384	1.18689	0.00109	0.00759

Table 4. Parameter estimates for the skewed *t* distribution.

	μ	δ	β	ν
AUD	0.00215	0.00654	−0.28196	3.04452
BRL	0.00152	0.00551	−0.20463	2.46025
CAD	0.00135	0.00513	−0.25383	3.83421
CHF	−0.00094	0.00567	0.22829	4.72209
EUR	0.00063	0.00506	−0.12117	4.19872
GBP	0.00100	0.00529	−0.21496	4.57395
JPY	−0.00025	0.00530	0.10035	4.42075
MXN	0.00195	0.00444	−0.48198	2.65491
TRY	0.00239	0.00655	−0.41112	3.36613

Table 5. Parameter estimates for the NIG distribution.

	α	β	δ	μ
AUD	72.01293	−10.19284	0.00729	0.00125
BRL	57.42940	−6.87905	0.00565	0.00082
CAD	127.45220	−11.53077	0.00628	0.00065
CHF	141.86882	6.88453	0.00762	−0.00014
EUR	138.78464	−4.90461	0.00646	0.00029
GBP	152.40781	−10.19284	0.00729	0.00125
JPY	143.68868	5.94505	0.00711	−0.00002
MXN	89.48977	−19.07877	0.00457	0.00082
TRY	86.53059	−14.18193	0.00741	0.00101

differences $(F_e(x) - \hat{F}(x))^2$, where the weights are chosen in such a way that the discrepancies in the tails are emphasized. The smaller the KS and AD test statistic, the better the fit.

The resulting KS and AD test statistics are given in tables 6 and 7, respectively. We have also included the corresponding statistics of the normal distribution for comparison purposes. The entries with $*$ correspond to the row minimum. We find that the non-normal models outperform the normal distribution when the comparison is done with both the KS test statistic and the AD test statistic. The comparison of the KS statistic among

Table 6. KS test statistics for the log-returns.

	GLD	S_U	Skewed t	NIG	Normal
AUD	0.016	0.015	0.015	0.014*	0.084
BRL	0.031	0.027*	0.032	0.031	0.112
CAD	0.026*	0.030	0.026*	0.027	0.068
CHF	0.017	0.016*	0.016*	0.016*	0.051
EUR	0.019	0.015*	0.019	0.017	0.054
GBP	0.017	0.022	0.017	0.016*	0.050
JPY	0.017*	0.026	0.019	0.017*	0.052
MXN	0.019	0.013*	0.014	0.019	0.094
TRY	0.023*	0.026	0.025	0.025	0.085

Table 7. AD test statistics for the log-returns.

	GLD	S_U	Skewed t	NIG	Normal
AUD	0.361	0.280	0.250*	0.422	27.321
BRL	0.884*	1.298	1.038	0.935	30.281
CAD	0.533	1.048	0.710	0.406*	12.576
CHF	0.387	0.399	0.267*	0.333	10.815
EUR	0.601	0.492	0.756	0.435*	10.276
GBP	0.511	1.195	0.474*	0.512	9.229
JPY	0.353	0.625	0.397	0.341*	9.549
MXN	0.805	0.436	0.280*	0.786	38.801
TRY	0.471*	0.891	0.539	0.473	20.554

non-normal models reveals no significant difference between the fits. However, the AD test statistic favours the Skewed t in the case of four currencies AUD, CHF, GBP and MXN, while the NIG is favoured in the case of CAD, EUR, JPY and TRY. GLD performs similar to NIG for TRY, while GLD outperforms other models for BRL.

4.2. *Risk estimation*

This section investigates the behaviour of the models at the tails using VaR and ES as risk measures. More specifically, we use the fitted distributions to determine the risk for long and short positions of each currency at levels $\alpha \in \{0.005, 0.01, 0.05, 0.95, 0.99, 0.995\}$. The first three levels are used to measure the risk of long positions, while the last three levels are used to measure the risk of short positions. We first compute in-sample VaR(α) levels in order to investigate the behaviour of the fitted models at the tails. To this end, we apply Kupiec likelihood ratio test (Kupiec 1995), which tests the hypothesis that the expected proportion of violations is equal to α. The likelihood ratio statistic is given by $2\log((\tau(\alpha)/n)^{\tau(\alpha)}(1-\tau(\alpha)/n)^{n-\tau(\alpha)}) - 2\log(\alpha^{\tau(\alpha)}(1-\alpha)^{n-\tau(\alpha)})$, where $\tau(\alpha)$ is the number of times the observed returns are above (for short positions) or below (for long positions) the theoretical VaR value, and n is the length of the data. Under the null hypothesis, this statistic is distributed as a χ^2 distribution with one degree of freedom.

Table 8 presents the p-values of the likelihood ratio statistic for both short and long positions. Given that we use a 5% level for the test, the normal model is rejected 30 times demonstrating the poor performance of the normal at the tails (rejected values are indicated in italic). The S_U is rejected 10 times, while the Skewed t is rejected only once. Finally, for both the NIG and the GLD the p-values are always greater than 0.05, and these two models are never rejected. Thus, in terms of the in-sample VaR(α) performance at different levels of α, the NIG, the GLD and the Skewed t yield good performance for all of the currencies.

We now compute the ES value at each α level for each currency. To backtest the predicted ES value, we use the measure in Embrechts *et al*. (2004). In particular, we let R_t represent the log-return value at time t; $ES_p(\alpha)$ the predicted ES value at level α; and compute the difference between the R_t and $ES_p(\alpha)$ at each α level; i.e. $\kappa(\alpha) = R_t - ES_p(\alpha)$. We further denote $\tilde{F}(\alpha)$ the empirical α quantile of $\kappa(\alpha)$; $\delta(\alpha)$ be the number of times $\kappa(\alpha)$ is less than (long positions) or greater than (short positions) $\tilde{F}(\alpha)$; and $\zeta(\alpha)$ the set of log-return values for which this happens. Similarly, $z(\alpha)$ denotes the set of log-return values for which a violation of VaR(α) occurs. We then compute $T_1(\alpha) = \sum_{t\in z(\alpha)} \kappa_t(\alpha)/\tau(\alpha)$ and $T_2(\alpha) = \sum_{t\in\zeta(\alpha)} \kappa_t(\alpha)/\delta(\alpha)$. Finally, the measure is given by $T(\alpha) = (|T_1(\alpha)| + |T_2(\alpha)|)/2$, where $|\cdot|$ indicates the absolute value. The lower the value of $T(\alpha)$, the better the estimated ES.

Table 9 presents $T(\alpha)$ values for each currency at each α level. The column minimum(s) is (are) highlighted with $*$. The normal distribution gives lower values than the other distributions (or equal values) 2 times, while the S_U gives lower values than the other distributions (or equal values) 16 times. The Skewed t and the GLD give lower values than the other distributions (or equal values) 30 times and the NIG gives lower values than the other distributions (or equal values) 29 times. Hence, the Skewed t, the GLD, and the NIG outperform the S_U for the prediction of the ES. This suggests that the GLD can be used as an alternative to the NIG and the Skewed t.

Table 8. p-values from the Kupiec test (in-sample VaR).

	Significance Levels					
Currency/ Method	0.005	0.01	0.05	0.95	0.99	0.995
AUD						
GLD	0.950	0.670	0.704	0.977	0.560	0.681
S_U	0.455	0.929	0.977	0.751	0.560	0.455
Skewed *t*	0.950	0.670	0.884	0.510	0.560	0.763
NIG	0.681	0.670	0.274	0.884	0.560	0.681
Normal	*0.000*	*0.049*	*0.028*	*0.002*	0.128	*0.010*
BRL						
GLD	0.950	0.868	0.666	0.704	0.488	0.681
S_U	0.455	0.560	0.666	0.704	0.488	0.681
Skewed *t*	0.950	0.868	0.510	0.704	0.212	0.763
NIG	0.681	0.929	0.977	0.224	0.334	0.681
Normal	*0.000*	*0.001*	0.274	*0.002*	0.410	0.090
CAD						
GLD	0.130	0.488	0.440	0.510	0.488	0.130
S_U	0.130	*0.005*	0.931	0.463	*0.005*	*0.013*
Skewed *t*	0.130	0.670	0.440	0.376	0.488	0.130
NIG	0.278	0.670	0.510	0.510	0.488	0.130
Normal	*0.000*	*0.001*	0.585	0.114	0.196	0.090
CHF						
GLD	0.763	0.670	0.224	0.376	0.066	0.130
S_U	0.455	0.670	0.931	0.931	*0.014*	0.130
Skewed *t*	0.681	0.670	0.884	0.376	0.066	0.130
NIG	0.763	0.670	0.224	0.510	0.066	0.130
Normal	0.090	0.868	*0.001*	0.538	0.560	0.455
EUR						
GLD	0.278	0.868	0.376	0.884	0.488	0.763
S_U	*0.013*	0.212	0.931	0.840	0.334	0.278
Skewed *t*	*0.013*	0.868	0.376	0.510	0.488	0.763
NIG	0.130	0.670	0.840	0.884	0.488	0.763
Normal	*0.001*	*0.005*	0.704	0.463	0.081	*0.023*
GBP						
GLD	0.950	0.735	0.114	0.088	0.735	0.495
S_U	*0.000*	*0.001*	0.977	0.931	*0.029*	*0.004*
Skewed *t*	0.950	0.560	0.114	0.224	0.735	0.495
NIG	0.455	0.735	0.068	0.088	0.735	0.495
Normal	*0.000*	*0.001*	0.068	*0.007*	0.289	*0.010*
JPY						
GLD	0.284	0.735	0.931	0.619	0.334	0.681
S_U	0.284	0.289	0.931	0.884	0.670	0.681
Skewed *t*	0.284	0.735	0.840	0.884	0.488	0.681
NIG	0.284	0.735	0.538	0.274	0.334	0.681
Normal	*0.004*	0.081	0.068	*0.038*	0.196	*0.023*
MXN						
GLD	0.950	0.670	0.931	0.393	0.128	0.090
S_U	0.284	0.929	0.884	0.931	0.081	*0.000*
Skewed *t*	0.495	0.670	0.619	0.884	0.128	0.166
NIG	0.284	0.670	0.463	0.274	0.128	0.470
Normal	*0.000*	*0.005*	*0.038*	*0.000*	0.128	*0.000*
TRY						
GLD	0.763	0.670	0.510	0.393	0.929	0.495
S_U	0.950	0.670	0.666	0.884	0.868	0.278
Skewed *t*	0.763	0.670	0.510	0.666	0.929	0.495
NIG	0.950	0.670	0.977	0.330	0.929	0.495
Normal	*0.000*	*0.005*	0.619	*0.000*	0.289	*0.023*

Table 9. Backtest measure of the ES predictions.

	Significance Levels					
Currency/ Method	0.005	0.01	0.05	0.95	0.99	0.995
AUD						
GLD	0.006	0.004	0.000*	0.000*	0.001*	0.002*
S_U	0.011	0.008	0.001	0.000*	0.001*	0.002*
Skewed *t*	0.003*	0.002*	0.000*	0.000*	0.002	0.003
NIG	0.013	0.008	0.001	0.000*	0.002	0.004
Normal	0.024	0.018	0.005	0.003	0.008	0.012
BRL						
GLD	0.012	0.006	0.001*	0.000*	0.002	0.001*
S_U	0.002*	0.001*	0.001*	0.000*	0.002	0.002
Skewed *t*	0.016	0.007	0.001*	0.001	0.001*	0.001*
NIG	0.004	0.003	0.001*	0.001	0.004	0.004
Normal	0.020	0.016	0.006	0.003	0.011	0.016
CAD						
GLD	0.004	0.003	0.001	0.000*	0.001*	0.003*
S_U	0.009	0.005	0.002	0.002	0.004	0.007
Skewed *t*	0.006	0.004	0.001	0.001	0.002	0.003*
NIG	0.002*	0.002*	0.000*	0.000*	0.001*	0.003*
Normal	0.005	0.004	0.002	0.001	0.003	0.005
CHF						
GLD	0.010	0.005*	0.000*	0.001*	0.002*	0.004*
S_U	0.010	0.007	0.001	0.001*	0.003	0.004*
Skewed *t*	0.008*	0.005*	0.000*	0.001*	0.002*	0.004*
NIG	0.011	0.005*	0.000*	0.001*	0.002*	0.004*
Normal	0.013	0.009	0.002	0.001*	0.004	0.007
EUR						
GLD	0.004	0.002*	0.001*	0.000*	0.001*	0.002
S_U	0.004	0.004	0.002	0.001	0.001*	0.002
Skewed *t*	0.004	0.004	0.001*	0.000*	0.001*	0.001*
NIG	0.002*	0.002*	0.001*	0.000*	0.001*	0.003
Normal	0.004	0.003	0.001*	0.001	0.005	0.007
GBP						
GLD	0.000*	0.000*	0.001*	0.001	0.001*	0.005
S_U	0.006	0.005	0.002	0.002	0.005	0.007
Skewed *t*	0.001	0.001	0.001*	0.000*	0.001*	0.003*
NIG	0.001	0.001	0.001*	0.001	0.002	0.005
Normal	0.007	0.005	0.002	0.001	0.005	0.007
JPY						
GLD	0.001	0.000*	0.000*	0.000*	0.002*	0.003
S_U	0.000*	0.001	0.000*	0.000*	0.003	0.005
Skewed *t*	0.003	0.001	0.000*	0.000*	0.002*	0.002*
NIG	0.000*	0.000*	0.000*	0.000*	0.003	0.005
Normal	0.005	0.004	0.002	0.002	0.007	0.010
MXN						
GLD	0.008	0.003*	0.001	0.001*	0.004	0.004
S_U	0.004	0.004	0.001	0.003	0.008	0.008
Skewed *t*	0.009	0.005	0.002	0.002	0.002*	0.002*
NIG	0.002*	0.003*	0.000*	0.001*	0.005	0.006
Normal	0.016	0.012	0.004	0.003	0.009	0.012
TRY						
GLD	0.003	0.002	0.001	0.000*	0.002	0.005
S_U	0.001*	0.001*	0.001	0.001	0.000*	0.003
Skewed *t*	0.004	0.002	0.001	0.000*	0.001	0.002*
NIG	0.005	0.003	0.000*	0.000*	0.002	0.006
Normal	0.016	0.012	0.004	0.002	0.006	0.009

5. Conclusion

This paper investigates the ability of the GLD in fitting the daily rates of nine currencies against the S_U, the Skewed *t*, and the NIG. We compare the overall fit of the models using the KS test statistic, AD test statistic and visual plots. We further investigate the behaviour of the models at the tails using the VaR and ES as risk measures.

Our results suggest that in terms of overall fit all methods perform similarly to each other. However, the NIG and the Skewed *t* give lower values of the AD statistic suggesting that they perform slightly better at the tails. This observation is supported when the comparison of the models are performed using VaR and ES as risk measures. In particular, in terms of VaR and ES performance we find that the NIG, the Skewed *t*, and the GLD do a good job in capturing tail behaviour of the exchange rates.

The observation that the GLD performs similarly to the NIG and the Skewed *t* suggests the GLD as an alternative for measuring fat-tail risk. Furthermore, GLD is very convenient in generating random variates in Monte Carlo simulations, which is widely used in pricing derivative securities and in risk management. The derivative security could be an exotic currency option whose complex payoff structure does not allow analytical solutions or it could be a mortgage bond where the option adjusted spread needs to be calculated for a given price† which requires simulating prepayment options under different interest rate and market scenarios. The need to price a derivative as fast and as accurately as possible is a challenge, wherein a flexible distribution like GLD can become handy. In particular, the percentile function representation of GLD makes it more convenient to generate random variates from GLD in Monte Carlo simulations; e.g. the generation of random variates from GLD is approximately 3 times faster than the generation of random variates from Skewed *t* and 16 times faster than the generation of random variates from NIG.

For risk management purposes, risk measures such as VaR and ES will typically be estimated for large portfolios that can include derivatives whose non-linear relation to underlying securities will make such estimations to be obtained analytically very difficult. The other alternative to analytical methods known as historical simulation will suffer from gaps in the data, especially in the tails (Glasserman 2000, Malz 2011). Hence, a Monte Carlo simulation will be the only viable option and GLD will be favourable with its computational advantages. More specifically, in order to simulate the losses of a portfolio of assets with known correlations, a copula framework can easily be utilized to generate correlated fat-tailed marginals from GLD. The extension of this single security analysis to a portfolio of securities is the subject of future research.

Acknowledgements

The authors thank Murat Tinic for obtaining the data and for providing some of the parameter estimates. The authors also thank the referees for their helpful feedback to improve this article.

Funding

This work was supported in part by TÜBİTAK (Turkish Scientific and Technological Research Council) Grant [111M425].

†In mortgage bond pricing the prepayment option is important and usually either the market price is given and the option adjusted spread (OAS) is calculated or the price is derived given the OAS.

References

Akgiray, V. and Booth, G.G., Mixed diffusion-jump process modeling of exchange rate movements. *Rev. Econ. Stat.*, 1988, **70**, 631–637.

Anderson, T.W. and Darling, D.A., A test of goodness of fit. *J. Am. Stat. Assoc.*, 1954, **49**, 765–769.

Azzalini, A. and Capitanio, A., Distributions generated by perturbation of symmetry with emphasis on a multivariate skew *t* distribution. *J. Roy. Stat. Soc. B*, 2003, **65**, 579–602.

Barndorff-Nielsen, O.E. and Prause, K., Apparant scaling. *Finance Stoch.*, 2001, **5**, 103–113.

Bauwens, L. and Laurent, S., A new class of multivariate skew densities, with application to GARCH models. *J. Bus. Econ. Stat.*, 2005, **23**, 346–354.

Biller, B. and Corlu, C.G., Least-squares and Bayesian inferences for Johnson's S_L and S_B distributions. Working Paper, Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA, 2012.

Bolviken, E. and Benth, F.E., Quantification of risk in Norwegian stocks via the normal inverse Gaussian distribution. Proceedings of the AFIR 2000 Colloquium, Tromso, Norway, pp. 87–98, 2000.

Boothe, P. and Glassman, D., The statistical distribution of exchange rates: Empirical evidence and economic implications. *J. Int. Econ.*, 1987, **22**, 297–319.

Branco, M.D. and Dey, D.K., A general class of multivariate skew-elliptical distributions. *J. Multivar. Anal.*, 2001, **79**, 99–113.

Chakravant, I.M., Laha, R.G. and Roy, J., *Handbook of Methods of Applied Statistics*, Vol. I, pp. 392–394, 1967 (John Wiley and Sons: New York, NY).

Chalabi, Y., Scott, D. and Würtz, D., The generalized lambda distribution as an alternative to model financial returns. Working paper, Eidgenössische Technische Hochschule and University of Auckland, Zurich and Auckland, 2010.

Choi, P., Estimation of value at risk using Johnson S_U-normal distribution. Working paper, Texas A&M University, College Station, TX, 2001.

Choi, P. and Nam, K., Asymmetric and leptokurtic distribution for heteroscedastic asset returns: The S_U-normal distribution. *J. Emp. Finance*, 2008, **15**, 41–63.

Corlu, C.G. and Meterelliyoz, M., Estimating the parameters of the generalized lambda distribution: Which method performs best? *Commun. Stat-Simul. C*, 2014. doi:10.1080/03610918.2014.901355.

Corrado, C.J., Option pricing based on the generalized lambda distribution. *J. Futures Markets*, 2001, **21**, 213–236.

Eberlein, E. and Keller, U., Hyperbolic distributions in finance. *Bernoulli*, 1995, **1**, 281–299.

Embrechts, P., Kaufmann, R. and Patie, P., Strategic long-term financial risks: Single risk factors. *Comput. Optim. Appl.*, 2004, **32**, 61–90.

Fama, E., The behavior of stock-market prices. *J. Bus.*, 1965, **38**, 34–105.

Farber, A., Roll, R. and Solnik, B., An empirical study of risk under fixed and flexible exchange rates. In *Stabilization of the Domestic and International Economy* edited by K. Brunner and A. Meltzer, pp. 235–275, 1977 (North Holland: Amsterdam).

Fernandez, C. and Steel, M., On Bayesian modeling of fat tails and skewness. *J. Am. Stat. Assoc.*, 1998, **93**, 359–371.

Filliben, J.J., The probability plot correlation coefficient test for normality. *Technometrics*, 1975, **17**, 111–117.

Fournier, B., Rupin, N., Bigerelle, M., Najjar, D., Iost, A. and Wilcox, R., Estimating the parameters of a generalized lambda distribution. *Comput. Stat. Data Anal.*, 2007, **51**, 2813–2835.

Freimer, M., Mudholkar, G., Kollia, G. and Lin, C., A study of the generalized Tukey lambda family. *Commun. Stat.-Theory Meth.*, 1988, **17**, 3547–3567.

Glasserman, P., *Monte Carlo Methods in Financial Engineering*, 2000 (Springer: New York, NY).

Gurrola, P., Capturing fat-tail risk in exchange rate returns using S_U curves: A comparison with the normal mixture and skewed Student distributions. *J. Risk*, 2008, **10**, 73–100.

Hansen, B., Autoregressive conditional density estimation. *Int. Econ. Rev.*, 1994, **35**, 705–730.

Hastings, J.C., Mosteller, F., Tukey, J. and Windsor, C., Low moments for small samples: A comparative study of order statistics. *Ann. Stat.*, 1947, **18**, 413–426.

Huisman, R., Koedjik, K., Kool, C. and Palm, F., The fat-tailness of FX returns reconsidered. *De Econ.*, 2002, **150**, 299–312.

Hull, J. and White, A., Value at risk when daily changes in market variables are not normally distributed. *J. Derivatives*, 1998, **5**, 9–19.

Johnson, N.L., Systems of frequency curves generated by methods of translation. *Biometrika*, 1949, **36**, 149–176.

Johnson, K. and Scott, E., The statistical distribution of daily exchange rate price changes: Dependent and independent models. *J. Financ. Strategic Decisions*, 1999, **12**, 39–49.

Joiner, B.L. and Rosenblatt, J.R., Some properties of the range in samples from Tukey's symmetric lambda distributions. *J. Am. Stat. Assoc.*, 1971, **66**, 394–399.

Jones, M.C. and Faddy, M.J., A skew extension of the *t* distribution with applications. *J. Roy. Stat. Soc. Ser. B*, 2003, **65**, 159–174.

Kupiec, P.H., Techniques for verifying the accuracy of risk measurement models. *J. Derivatives*, 1995, **2**, 73–84.

Lee, P.P., The generalized lambda distribution applied to spot exchange rates. PhD Dissertation, Carnegie Mellon University, 2003.

Lillestol, J., Risk analysis and the NIG distribution. *J. Risk*, 2000, **2**, 41–56.

Malz, A.M., *Financial Risk Management: Models, History, and Institutions*, 2011 (John Wiley and Sons: Hoboken, NJ).

Mandelbrot, B.B., The variation of certain speculative prices. *J. Bus.*, 1963, **36**, 394–419.

McFarland, J.W., Pettit, R.R. and Sung, S.K., The distribution of foreign exchange price changes: Trading day effects and risk measurement. *J. Finance*, 1982, **37**, 693–715.

Patton, A., On the out-of-sample importance of skewness and asymmetric dependence for asset allocation. *J. Financ. Econ.*, 2004, **2**, 130–168.

Pfaff, B., QRM: Provides R-language code to examine quantitative risk management concepts. R package version 0.4–7, 2012.

Prause, K., How to use NIG laws to measure market risk FDM. Preprint 65, University of Freiburg, 1999.

Ramberg, J.S. and Schmeiser, B.W., An approximate method for generating asymmetric random variables. *Commun. ACM*, 1974, **17**, 78–82.

Sahu, S.K., Dey, D.K. and Branco, M.D., A new class of multivariate skew distributions with applications to Bayesian regression models. *Canadian J. Stat.*, 2003, **31**, 129–150.

Simonato, J.-G., The performance of Johnson distributions for computing value at risk and expected shortfall. *J. Derivatives*, 2011, **19**, 7–24.

Su, S., Numerical maximum log likelihood estimation for generalized lambda distributions. *Comput. Stat. Data Anal.*, 2007, **51**, 3983–3998.

Tarsitano, A., Fitting the generalized lambda distribution to income data. COMPSTAT 2004-Proceedings in Computational Statistics, 16th Symposium Held in Prague, Czech Republic, 2004.

Tucker, A.L. and Pond, L., The probability distribution of foreign exchange price changes: Tests of candidate processes. *Rev. Econ. Stat.*, 1988, **70**, 638–647.

Tuenter, H.J., An algorithm to determine the parameters of the S_U curves in the Johnson system of probability distributions by moment matching. *J. Stat. Comput. Simul.*, 2001, **70**, 325–347.

Venter, J.H. and de Jongh, P.J., Risk estimation using the normal inverse Gaussian distribution. *J. Risk*, 2002, **4**, 1–24.

Westerfield, J.M., An examination of foreign exchange risk under fixed and floating rate regimes. *J. Int. Econ.*, 1977, **7**, 181–200.