Empir Econ (2015) 48:1691–1712
DOI 10.1007/s00181-014-0845-3

GARCH modeling of five popular commodities

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Received: 24 June 2013 / Accepted: 18 April 2014 / Published online: 22 July 2014

Abstract Flexible models for the innovation process of GARCH models have been limited. Here, we show the flexibility of two recently proposed distributions due to Zhu and Zinde-Walsh (J Econom 148:86–99, 2009) and Zhu and Galbraith (J Econom 157:297–305, 2010) by means of GARCH modeling of five popular commodities. The five commodities considered are Cocoa bean, Brent crude oil, West Texas intermediate crude oil, Gold and Silver. For each commodity, one of the two models due to Zhu and Zinde-Walsh (2009) and Zhu and Galbraith (2010) is shown to perform better than those commonly known.

Keywords Cocoa bean · GARCH models · Gold · Oil · Silver

1 Introduction

Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models have been the most popular models for financial time series. Their applications to finance and related areas are too numerous to mention. Some recent applications have included analysis of the daily returns in Istanbul Stock Exchange (Bildirici and Ersin 2009); Lithuanian stock market analysis (Teresiene 2009); the relationship between the Vietnam stock market and its major trading partners (Chang and Su 2010); forecasting financial volatility of the Athens stock exchange daily returns (Drakos et al. 2010); the effect of exchange-rate uncertainty on unemployment in three developing Asian countries (Chang and Shen 2011); electricity price forecasting (Santos Coelhoa and Santos 2011); arbitrage behavior in the exchange rates of Taiwan and Japan (Lee and Chiu 2011); interdependence between the Slovenian and European stock markets

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(Dajcman and Festic 2012); analyzing effects of gasoline price and miles traveled on fatal crashes involving intoxicated drivers in Texas (Ye et al. 2012); and risk estimation to the capital market in Romania (Acatrinei et al. 2013).

A GARCH model is composed of two components: the volatility component and the innovation component. The simplest and the most commonly used model for volatility is of the order of $(1, 1)$. The innovation is commonly assumed to come from the Gaussian distribution, the Student's t distribution or some skewed extension of these distributions.

Available models for innovation have been limited. This is partly because of the lack of freely available software for fitting of GARCH models. The R (R Development Core Team 2013) contributed package fGarch due to Wuertz and Chalabi (2013) has been the most popular software for fitting of GARCH models. But this software limits the models for innovation to be one of the following: the Gaussian (NORM) distribution due to de Moivre (1738) and Gauss (1809); the skewed Gaussian (SNORM) distribution due to Azzalini (1985); the Student's t (ST) distribution due to Gosset (1908); the skewed Student's t (SST0) distribution due to Fernandez and Steel (1998); the generalized error (GE) distribution due to Subbotin (1923); the skewed generalized error (SGE) distribution due to Theodossiou (1998); and the standardized normal inverse Gaussian (NIG) distribution due to Barndorff-Nielsen (1977).

The aim of this paper was to introduce two models for innovations and to illustrate their flexibility over the ones implemented in the package fGarch. The models are based on two recently proposed distributions due to Zhu and Zinde-Walsh (2009) and Zhu and Galbraith (2010).

Zhu and Zinde-Walsh (2009) proposed the asymmetric exponential power (AEP) distribution, the most general form of the NORM distribution known to date. Its probability density function (PDF) is

$$f(x) = C \begin{cases} \exp\left\{ -\frac{1}{p_1} \left[\frac{\mu - x}{2\alpha} \right]^{p_1} \right\}, & \text{if } x \leq \mu, \\ \exp\left\{ -\frac{1}{p_2} \left[\frac{x - \mu}{2(1-\alpha)} \right]^{p_2} \right\}, & \text{if } x > \mu \end{cases} \tag{1}$$

for $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\alpha > 0$, $p_1 > 0$ and $p_2 > 0$, where C is given by

$$C = \frac{1}{2\alpha A_0(p_1) + 2(1-\alpha) A_0(p_2)}, \tag{2}$$

where

$$A_0(x) = x^{(1/x)-1} \Gamma\left(\frac{1}{x}\right).$$

Here, p_1 and p_2 are shape parameters, α is a scale parameter, and μ is a location parameter.

Zhu and Galbraith (2010) proposed the asymmetric Student's t (AST) distribution, the most general form of the ST distribution known to date. Its (PDF) is

$$f(x)=\begin{cases}\dfrac{\alpha}{\alpha^{*}}K\left(\nu_1\right)\left\{1+\dfrac{1}{\nu_1}\left[\dfrac{x-\mu}{2\alpha^{*}}\right]^2\right\}^{-\frac{\nu_1+1}{2}}, & \text{if } x\leq\mu,\\[2ex] \dfrac{1-\alpha}{1-\alpha^{*}}K\left(\nu_2\right)\left\{1+\dfrac{1}{\nu_2}\left[\dfrac{x-\mu}{2\left(1-\alpha^{*}\right)}\right]^2\right\}^{-\frac{\nu_2+1}{2}}, & \text{if } x>\mu\end{cases}\tag{3}$$

for $-\infty<x<\infty$, $-\infty<\mu<\infty$, $0<\alpha<1$, $\nu_1>0$ and $\nu_2>0$, where

$$\alpha^{*}=\frac{\alpha K\left(\nu_1\right)}{\alpha K\left(\nu_1\right)+(1-\alpha)K\left(\nu_2\right)}.$$

Here, ν_1 and ν_2 are degree of freedom parameters, α is a scale parameter, and μ is a location parameter.

The contents of this paper are organized as follows. In Sect. 2, we describe the data used in the paper. The data are stock market price returns on five popular commodities. In Sect. 3, we describe eleven models for innovation including the two stated above. For each model, we give explicit expressions for the value at risk and expected shortfall. However, not all of these expressions are new. For example, those for the AEP and AST distributions can be found in Zhu and Zinde-Walsh (2009) and Zhu and Galbraith (2010). The expressions are given for completeness. In Sect. 4, GARCH models with each of the eleven innovation distributions are fitted to the data described in Sect. 2. It is shown that the models based on (1) and (3) outperform all of the others. The computer software used for the results in Sect. 4 can be obtained from the corresponding author, e-mail: mbbsssn2@manchester.ac.uk. The relationship of the results in Sect. 4 to known work is described in Sect. 5. Some conclusions are noted in Sect. 6.

2 Data

The data we consider are daily stock market price returns of five popular commodities: Cocoa bean, Brent crude oil, West Texas intermediate crude oil, and Gold and Silver. The data cover the period from the 12th of March 1993 to the 13th of March 2013. The data were obtained from the database Datastream.

Cocoa beans are a product of the fruit from the plant Theobroma cacao. This plant is commonly seen in areas of Africa and Asia. Cocoa beans are dried dull red in appearance. The flavor and aroma of the cocoa bean is developed through the fermentation process lasting several days. Cocoa beans are most commonly used for processed foods and chocolate (Lecumberri et al. 2007, p. 948).

The significance of oil is great, particularly in the production of petrol used in motor vehicles. Brent Crude oil can usually be found being refined and consumed in great quantities in North West Europe. Its properties, for example being a light combination of crude oils from numerous fields from the North Sea, make it an excellent choice for producing gasoline and middle distillates. Brent Crude has an API gravity of 38.8, indicating that it is less sweet compared with West Texas Intermediate Crude oil (Speight 2011).

West Texas Intermediate is also a light and sweet crude oil like Brent Crude. Its high quality means that it is suitable in the production of large quantities of gasoline. It has an API gravity of 39.6 and contains approximately 0.24 % sulfur. West Texas Intermediate is refined in the United States—the country that consumes the greatest quantity of gasoline (Speight 2011).

Gold (chemical symbol "Au") is a metal with many unique properties like having a bright metallic yellow appearance, an excellent resistance to corrosion, a considerable malleability, and a high density. These properties make gold very suitable for the production of Jewelry (Corti and Holliday 2010, p. 13).

Silver (chemical symbol "Ag") is one of the softer metals, but one which can easily be shaped. Because of this, silver is usually hardened by combining with other metals. With silver taking a bright gray and white appearance, it is commonly used in the production of mirrors, cutlery, and jewelery. Using silver in these items gives a brighter more sparkling look as silver is known for reflecting light better than many other metals (Belval 2007, pp. 14–18).

Cocoa is an extremely important exporting commodity for West African nations like Ghana and Ivory Coast. These countries account for more than 70 % of the world's cocoa. For Ghana, cocoa "contributes 25 % annually of the total foreign exchange earnings but also being the source of livelihoods for many rural farmers and the related actors in the value chain" (Essegbey and Ofori-Gyamfi 2012).

Crude oil (Brent crude oil and West Texas intermediate crude oil) is a central source of energy supply and is the driving force behind the emerging economies of China, India, Russia, and Brazil.

Gold is an important commodity for many economies as it acts as a significant source of exports and foreign exchange earnings. In 2012, gold exports "were 36 % of all Tanzanian merchandize exports, 26 % of exports in both Ghana and Papua New Guinea and 21 % of Peruvian exports" (World Gold Council 2012, p. 4). The process of mining for gold provides employment for many developing economies: "The total direct employment in gold mining across the 15 largest gold mining countries is estimated to be 527,900 in 2012. Three countries stand out: South Africa has an estimated 145,600 gold mining employees, Russia has an estimated 138,000 gold mining employees, and China has an estimated 98,200 gold mining employees" (World Gold Council 2012, p. 16).

Silver is a major foreign income earner for Mexico. This country recorded an output of around 20 % of the world's silver in 2011. In the financial market, Silver is becoming a major investment commodity.

The histograms of the five stock price returns are shown in Fig. 1. It appears visually that the returns of each of the commodities are symmetrical about zero.

Some basic statistics of the stock price returns are summarized in Table 1. The basic statistics summarized are minimum, first quartile, median, mean, third quartile, maximum, standard deviation, coefficient of variation, skewness, kurtosis, inter-quartile range, and range.

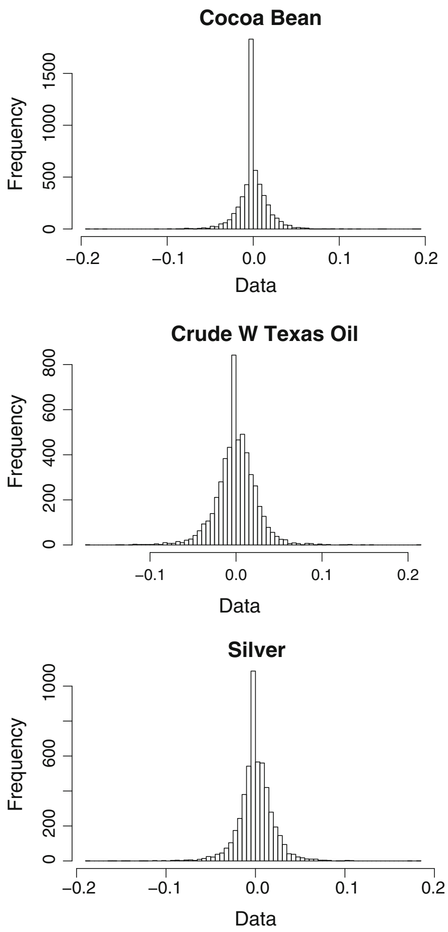

Fig. 1 Histograms of the five stock price returns

The minimum value for each commodity is negative. It is smallest for Cocoa bean and largest for Gold. The first quartile value for each commodity is also negative. It is smallest for West Texas intermediate crude oil and largest for Gold. The median is exactly zero for Cocoa bean, Gold, and Silver. The median is largest for West Texas intermediate crude oil. The mean is smallest for Cocoa bean and largest for Silver. The third quartile is smallest for Gold and largest for West Texas intermediate crude oil. The maximum is smallest for Gold and largest for West Texas intermediate crude oil. The standard deviation is smallest for Gold and largest for West Texas intermediate crude oil.

The coefficient of variation is highest for Cocoa bean, followed by Brent crude oil, Silver, West Texas intermediate crude oil, and Gold in that order. This is an indication that Cocoa bean returns have been significantly more volatile than other commodities, while Gold returns have remained rather more stable compared to other commodities within the period under study.

Table 1 Summary statistics of stock price returns on the five commodities

	Cocoa bean	Brent crude oil	West Texas intermediate crude oil	Gold	Silver
Min	-1.928×10^{-1}	-1.363×10^{-1}	-1.722×10^{-1}	-7.143×10^{-2}	-1.869×10^{-1}
Q1	-6.094×10^{-3}	-1.097×10^{-2}	-1.147×10^{-2}	-3.856×10^{-3}	-8.609×10^{-3}
Median	0	3.335×10^{-4}	5.863×10^{-4}	0	0
Mean	1.528×10^{-4}	3.352×10^{-4}	2.897×10^{-4}	3.023×10^{-4}	3.985×10^{-4}
Q3	6.431×10^{-3}	1.245×10^{-2}	1.264×10^{-2}	4.791×10^{-3}	9.989×10^{-3}
Max	1.938×10^{-1}	1.35×10^{-1}	2.128×10^{-1}	7.382×10^{-2}	1.828×10^{-1}
SD	1.787×10^{-2}	2.154×10^{-2}	2.379×10^{-2}	1.008×10^{-2}	2.017×10^{-2}
CV	117.007	64.280	41.632	33.346	50.610
Skewness	4.248×10^{-2}	-9.604×10^{-2}	-4.463×10^{-2}	-1.485×10^{-1}	-3.669×10^{-1}
Kurtosis	19.949	6.019	8.411	9.277	12.352
IQR	1.253×10^{-2}	2.342×10^{-2}	2.411×10^{-2}	8.647×10^{-3}	1.860×10^{-2}
Range	3.866×10^{-1}	2.713×10^{-1}	3.85×10^{-1}	1.453×10^{-1}	3.697×10^{-1}

The Cocoa bean price returns are positively skewed. The remaining price returns (Brent crude oil, West Texas intermediate crude oil, Gold and Silver) are negatively skewed. The smallest of the negative skewness is for West Texas intermediate crude oil. The largest is for Silver. This matches up with the fact that most financial data are negatively skewed. Campbell and Hentschel (1992) explain negative skewness of financial data as “large negative stock returns are more common than large positive ones, so stock returns are negatively skewed . . . this shows up clearly in the pattern of extreme moves in stock prices in the postwar period”. See also Mandelbrot (1963) and Fama (1965).

Each kurtosis value is significantly greater than three, the kurtosis value corresponding to the normal distribution. The smallest kurtosis is for Brent crude oil. The largest is for Cocoa bean. This matches up with the fact that most financial data have excess kurtosis. Campbell and Hentschel (1992) explain excess kurtosis of financial data as “extreme stock market movements are more common than would be expected if stock returns were drawn from a normal distribution, so stock returns have excess kurtosis. This is not just the result of changing volatility, because excess kurtosis remains after one normalizes returns by their estimated conditional standard deviations”. See also Mandelbrot (1963) and Fama (1965).

The inter-quartile range is smallest for Gold and largest for West Texas intermediate crude oil. The range is smallest for Gold and largest for Cocoa bean.

Normality of stock price returns for each commodity was tested using the Anderson–Darling test (Anderson and Darling 1954), the Cramer–von Mises test, the Kolmogorov–Smirnov test, the Pearson chi-square test, the Jarque–Bera test (Jarque and Bera 1980), the Geary test (Geary 1947), and the data-driven smooth test. The tests showed that none of the data sets on stock price returns followed the normal distribution.

3 The GARCH model and properties under different distributions

GARCH(1, 1) is a popular time series model for weakly stationary financial data. It can be specified by

$$X_t = \sigma_t Z_t, \tag{4}$$

where $\{X_t\}$ is the observed financial data, $\{\sigma_t\}$ is a *volatility process* specified by

$$\sigma_t^2 = \omega + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

and $\{Z_t\}$ is an *innovation process*.

We consider eleven different distributions for Z_t: the NORM distribution, the SNORM distribution, the ST distribution, the SST0 distribution, the GE distribution, the SGE distribution, the NIG distribution, the AEP distribution, the Skewed Exponential Power (SEP) distribution (the particular case of the AEP distribution for $p_1 = p_2$), the AST distribution, and the Skewed Student's t (SST) distribution (the particular case of the AST distribution for $\nu_1 = \nu_2$).

The first six distributions are the commonly used models for the innovation process. They are implemented in standard computer packages for GARCH modeling. See, for example, the R (R Development Core Team 2013) contributed package fGarch due to Wuertz and Chalabi (2013). The last four distributions are relatively new. We are not aware of any computer package that has implemented these distributions as possible innovation models.

For each distribution for Z_t, we give explicit expressions for $\mathrm{VaR}_p(Z_t)$ and $\mathrm{ES}_p(Z_t)$.

3.1 SNORM distribution

If Z_t are independent and identical SNORM random variables with location parameter μ and skewness parameter λ, then

$$\mathrm{ES}_p(Z_t) = 2 \int_{-\infty}^{\mathrm{VaR}_p} x \phi(x-\mu) \Phi(\lambda(x-\mu))\, \mathrm{d}x,$$

where $\mathrm{VaR}_p(Z_t)$ is the root of

$$\Phi(x-\mu) - 2T(x-\mu, \lambda) = p,$$

where $T(h, a)$ is Owen's function defined in Owen (1980), $\phi(\cdot)$ is the (PDF) of a standard NORM random variable, and $\Phi(\cdot)$ is the cumulative distribution function (CDF) of a standard NORM random variable. The moments of Z_t can be found in Azzalini (1985).

3.2 NORM distribution

If Z_t are independent and identical NORM random variables with mean μ and unit variance, then the expressions for $\mathrm{VaR}_p(Z_t)$ and $\mathrm{ES}_p(Z_t)$ follow from those given in Sect. 3.1 by setting $\lambda = 0$.

3.3 SST0 distribution

If Z_t are independent and identical SST0 random variables with location parameter μ, skewness parameter λ, and degrees of freedom ν, then

$$\mathrm{VaR}_p(Z_t) = \begin{cases} \mu + \sqrt{\gamma^{-2}\nu\left[\left\{I^{-1}_{2\gamma p}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right\}^{-1} - 1\right]}, & \text{if } p \le 1/(2\gamma), \\ \mu + \sqrt{\gamma^{2}\nu\left[\left\{I^{-1}_{1+\gamma^{-2}-2\gamma^{-1}p}\left(\frac{\nu}{2}, \frac{1}{2}\right)\right\}^{-1} - 1\right]}, & \text{if } p > 1/(2\gamma), \end{cases}$$

$$\mathrm{ES}_p(Z_t) = \begin{cases} \mu p + \frac{\sqrt{\nu}\Gamma((\nu+1)/2)}{\gamma^2(1-\nu)\sqrt{\pi}\Gamma(\nu/2)}\left(1 + \frac{\gamma^2 \mathrm{VaR}^2}{\nu}\right)^{\frac{1-\nu}{2}}, & \text{if } \mathrm{VaR} \le \mu, \\ \mu p + \frac{\sqrt{\nu}\Gamma((\nu+1)/2)}{(1-\nu)\sqrt{\pi}\Gamma(\nu/2)}\left[\gamma^2\left(1 + \frac{\mathrm{VaR}^2}{\gamma^2\nu}\right)^{\frac{1-\nu}{2}} - \gamma^2 + \gamma^{-2}\right], & \text{if } \mathrm{VaR} > \mu, \end{cases}$$

where $I_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1}dt/B(a, b)$ is the incomplete beta function ratio and $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the beta function. The moments of Z_t can be found in Fernandez and Steel (1998).

3.4 ST distribution

If Z_t are independent and identical ST random variables with location parameter μ and degrees of freedom ν, then the expressions for $\mathrm{VaR}_p(Z_t)$ and $\mathrm{ES}_p(Z_t)$ follow from those given in Sect. 3.3 by setting $\gamma = 1$.

3.5 SGE distribution

If Z_t are independent and identical SGE random variables with location parameter μ, shape parameter k, and skewness parameter λ, then

$$\mathrm{VaR}_p(Z_t) = \begin{cases} \mu - \delta - (1+\lambda)\theta\left[Q^{-1}\left(\frac{1}{k}, \frac{2p}{1+\lambda}\right)\right]^{1/k}, & \text{if } p \le \frac{1+\lambda}{2}, \\ \mu - \delta + (1-\lambda)\theta\left[Q^{-1}\left(\frac{1}{k}, \frac{2(1-p)}{1-\lambda}\right)\right]^{1/k}, & \text{if } p > \frac{1+\lambda}{2}, \end{cases}$$

$$\mathrm{ES}_p(Z_t) = \begin{cases} -\frac{C(1+\lambda)^2\theta^2}{k}\Gamma\left(\frac{2}{k}, \frac{(\mu-\mathrm{VaR}-\delta)^2}{(1+\lambda)^k\theta^k}\right), & \text{if } \mathrm{VaR} \le \mu - \delta, \\ -\frac{C(1+\lambda)^2\theta^2}{k}\Gamma\left(\frac{2}{k}\right) + \frac{C(1-\lambda)^2\theta^2}{k}\gamma\left(\frac{2}{k}, \frac{(\mathrm{VaR}-\mu+\delta)^2}{(1-\lambda)^k\theta^k}\right), & \text{if } \mathrm{VaR} > \mu - \delta, \end{cases}$$

where $C = k / \{2\theta\Gamma(1/k)\}$, $\theta = \sqrt{\Gamma(1/k)/\Gamma(3/k)}/S(\lambda)$, $\delta = 2\lambda A/S(\lambda)$, $S(\lambda) = \sqrt{1 + 3\lambda^2 - 4A^2\lambda^2}$, $A = \Gamma(2/k)/\sqrt{\Gamma(1/k)\Gamma(3/k)}$, $Q(a, x) = \int_x^\infty t^{a-1} \exp(-t)\, \mathrm{d}t / \Gamma(a)$ is the regularized complementary incomplete gamma function, $\gamma(a, x) = \int_0^x t^{a-1} \exp(-t)\, \mathrm{d}t$ is the incomplete gamma function, and $\Gamma(a, x) = \int_x^\infty t^{a-1} \exp(-t)\, \mathrm{d}t$ is the complementary incomplete gamma function. The moments of Z_t can be found in Theodossiou (1998).

3.6 GE distribution

If Z_t are independent and identical GE random variables with location parameter μ and shape parameter k, then the expressions for $\mathrm{VaR}_p(Z_t)$ and $\mathrm{ES}_p(Z_t)$ follow from those given in Sect. 3.5 by setting $\lambda = 0$.

3.7 NIG distribution

If Z_t are independent and identical NIG random variables, then

$$\mathrm{ES}_p(Z_t) = \frac{\alpha}{\pi} \int_{-\infty}^{\mathrm{VaR}} \frac{K_1\left(\alpha\sqrt{1 + (x - \mu)^2}\right)}{\sqrt{1 + (x - \mu)^2}} \exp(\beta x + \gamma)\, \mathrm{d}x,$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$, $K_1(\cdot)$ is the modified Bessel function of the second kind of order one, and $\mathrm{VaR}_p(Z_t)$ is the root of

$$\int_{-\infty}^{x} \frac{K_1\left(\alpha\sqrt{1 + (y - \mu)^2}\right)}{\sqrt{1 + (y - \mu)^2}} \exp(\beta y + \gamma)\, \mathrm{d}y = p.$$

The moments of Z_t can be found in Barndorff-Nielsen (1977).

3.8 AEP distribution

If Z_t are independent and identical AEP random variables, then

$$\mathrm{VaR}_p(Z_t) = \begin{cases} \mu - 2\alpha^* \left[p_1 R^{-1}\left(\frac{1}{p_1}, 1 - \frac{p}{\alpha}\right)\right]^{1/p_1}, & \text{if } p \le \alpha, \\ \mu - 2(1 - \alpha^*) \left[p_2 R^{-1}\left(\frac{1}{p_2}, 1 - \frac{1-p}{1-\alpha}\right)\right]^{1/p_2}, & \text{if } p > \alpha, \end{cases}$$

$$\mathrm{ES}_p(Z_t) = \begin{cases} \mu p - 2\alpha^* C(p_1) \dfrac{1 - R\left(\frac{2}{p_1}, \frac{1}{p_1}\left|\frac{\mathrm{VaR}-\mu}{2\alpha^*}\right|^{p_1}\right)}{1 - R\left(\frac{1}{p_1}, \frac{1}{p_1}\left|\frac{\mathrm{VaR}-\mu}{2\alpha^*}\right|^{p_1}\right)}, & \text{if VaR} \le \mu, \\ \mu p - \dfrac{2\alpha\alpha^* C(p_1) - 2(1-\alpha)(1-\alpha^*)C(p_2) R\left(\frac{2}{p_2}, \frac{1}{p_2}\left|\frac{\mathrm{VaR}-\mu}{2(1-\alpha^*)}\right|^{p_2}\right)}{\alpha + (1-\alpha) R\left(\frac{1}{p_2}, \frac{1}{p_2}\left|\frac{\mathrm{VaR}-\mu}{2(1-\alpha^*)}\right|^{p_2}\right)}, & \text{if VaR} > \mu, \end{cases}$$

where $R(a, x) = \int_0^x t^{a-1} \exp(-t)\,\mathrm{d}t / \Gamma(a)$ is the regularized incomplete gamma function, $K(p) = 1/\left\{2p^{1/p}\Gamma(1+1/p)\right\}$, $\alpha^* = \alpha K(p_1) / \{\alpha K(p_1) + (1-\alpha) K(p_2)\}$, $B = \alpha K(p_1) + (1-\alpha) K(p_2)$, $H_r(p) = p^r \Gamma((r+1)/p) / \Gamma^{r+1}(1/p)$ and $C(p) = p^{1/p}\Gamma(2/p)/\Gamma(1/p)$. The moments of Z_t can be found in Zhu and Zinde-Walsh (2009).

3.9 SEP distribution

If Z_t are independent and identical SEP random variables, then the expressions for $\mathrm{VaR}_p(Z_t)$ and $\mathrm{ES}_p(Z_t)$ follow from those given in Sect. 3.8 by setting $p_1 = p_2$.

3.10 AST distribution

If Z_t are independent and identical AST random variables, then

$$\mathrm{VaR}_p(Z_t) = \mu + 2\alpha^* S_{\nu_1}^{-1}\left(\frac{\min(p,\alpha)}{2\alpha}\right) + 2\left(1-\alpha^*\right) S_{\nu_2}^{-1}\left(\frac{\max(p,\alpha)+1-2\alpha}{2(1-\alpha)}\right),$$

$$\mathrm{ES}_p(Z_t) = \mu p - \frac{4B}{p}\left[\frac{(\alpha^*)^2 \nu_1}{\nu_1 - 1}\left\{1 + \frac{1}{\nu_1}\left[\frac{\min(\mathrm{VaR}-\mu, 0)}{2\alpha^*}\right]^2\right\}^{\frac{1-\nu_1}{2}} - \frac{(1-\alpha^*)^2 \nu_2}{\nu_2 - 1}\right.$$
$$\left. + \frac{(1-\alpha^*)^2 \nu_2}{\nu_2 - 1}\left\{1 + \frac{1}{\nu_2}\left[\frac{\max(\mathrm{VaR}-\mu, 0)}{2(1-\alpha^*)}\right]^2\right\}^{\frac{1-\nu_2}{2}}\right],$$

where $S_\nu(\cdot)$ is the CDF of a ST random variable with ν degrees of freedom, $K(\nu) = \Gamma((\nu+1)/2) / \left\{\sqrt{\pi\nu}\Gamma(\nu/2)\right\}$, $\alpha^* = \alpha K(\nu_1) / \{\alpha K(\nu_1) + (1-\alpha) K(\nu_2)\}$, $B = \alpha K(\nu_1) + (1-\alpha) K(\nu_2)$, and $H_r(\nu) = \sqrt{\nu^r/\pi}\,\Gamma((r+1)/2)\,\Gamma((\nu-r)/2) / \Gamma(\nu/2)$. The moments of Z_t can be found in Zhu and Galbraith (2010).

3.11 SST distribution

If Z_t are independent and identical SST random variables, then the expressions for $\mathrm{VaR}_p(Z_t)$ and $\mathrm{ES}_p(Z_t)$ follow from those given in Sect. 3.10 by setting $\nu_1 = \nu_2$.

4 Results and discussion

All of the distributions in Sect. 3 were fitted to each of the data sets on stock price returns discussed in Sect. 2. The method of maximum likelihood was used for parameter estimation. The function optimize in R (R Development Core Team 2013) was used for maximizing the likelihood function.

Table 2 gives parameter estimates, log-likelihood values, and Akaike information criterion (AIC) values for models fitted to Cocoa bean price returns. Table 3 gives parameter estimates, log-likelihood values, and AIC values for models fitted to Brent crude oil price returns. Table 4 gives parameter estimates, log-likelihood values, and

Table 2 Fitted models and estimates for Cocoa bean price returns

Model	Parameter estimates	$-\log L$	AIC
NORM	$\widehat{\mu} = 1.396 \times 10^{-2}$	−13924.5	−27841.0
ST	$\widehat{\mu} = -6.272 \times 10^{-3}$, $\widehat{\nu} = 5.656$	−15133.3	−30256.7
SST	$\widehat{\mu} = 5.145 \times 10^{-3}$, $\widehat{\nu} = 5.622$, $\widehat{\alpha} = 4.949 \times 10^{-1}$	−15148.2	−30284.3
AST	$\widehat{\mu} = 5.145 \times 10^{-3}$, $\widehat{\nu_1} = 5.622$, $\widehat{\nu_2} = 1.827$, $\widehat{\alpha} = 4.949 \times 10^{-1}$	−15479.6	−30945.2
GE	$\widehat{\mu} = 3.141 \times 10^{-2}$, $\widehat{k} = 1.618$	−13965.8	−27921.6
SEP	$\widehat{\mu} = 7.895 \times 10^{-2}$, $\widehat{p} = 1.608$, $\widehat{\alpha} = 5.085 \times 10^{-1}$	−14557.4	−29102.8
AEP	$\widehat{\mu} = 3.186 \times 10^{-4}$, $\widehat{p_1} = 1.206$, $\widehat{p_2} = 1.697$, $\widehat{\alpha} = 4.386 \times 10^{-1}$	−15073.4	−30132.8
SNORM	$\widehat{\mu} = 2.793 \times 10^{-2}$, $\widehat{\lambda} = 1.039$	−13929.8	−27849.6
SGE	$\widehat{\mu} = 1.569 \times 10^{-2}$, $\widehat{\lambda} = 9.379 \times 10^{-1}$, $\widehat{k} = 1.257$	−15055.6	−30099.1
SST0	$\widehat{\mu} = 2.236 \times 10^{-2}$, $\widehat{\gamma} = 9.675 \times 10^{-1}$, $\widehat{\nu} = 7.713$	−15141.8	−30271.5
NIG	$\widehat{\mu} = 1.933 \times 10^{-2}$, $\widehat{\alpha} = 2.122$, $\widehat{\beta} = -9.602 \times 10^{-2}$	−14707.6	−29403.1

Table 3 Fitted models and estimates for Brent crude oil price returns

Model	Parameter estimates	$-\log L$	AIC
NORM	$\widehat{\mu} = 2.004 \times 10^{-2}$	−13004.1	−26000.2
ST	$\widehat{\mu} = 3.211 \times 10^{-2}$, $\widehat{\nu} = 1.024 \times 10^{1}$	−13077.7	−26145.5
SST	$\widehat{\mu} = 1.128 \times 10^{-1}$, $\widehat{\nu} = 1.036 \times 10^{1}$, $\widehat{\alpha} = 5.277 \times 10^{-1}$	−13082.2	−26152.5
AST	$\widehat{\mu} = 8.857 \times 10^{-2}$, $\widehat{\nu_1} = 9.181$, $\widehat{\nu_2} = 1.197 \times 10^{1}$, $\widehat{\alpha} = 5.187 \times 10^{-1}$	−13082.6	−26151.1
GE	$\widehat{\mu} = 2.909 \times 10^{-2}$, $\widehat{k} = 1.501$	−13088.4	−26166.7
SEP	$\widehat{\mu} = 8.507 \times 10^{-2}$, $\widehat{p} = 1.542$, $\widehat{\alpha} = 5.085 \times 10^{-1}$	−13095.5	−26178.9
AEP	$\widehat{\mu} = 3.349 \times 10^{-4}$, $\widehat{p_1} = 1.319$, $\widehat{p_2} = 1.562$, $\widehat{\alpha} = 4.817 \times 10^{-1}$,	−13110.5	−26206.9
SNORM	$\widehat{\mu} = 1.581 \times 10^{-2}$, $\widehat{\lambda} = 9.449 \times 10^{-1}$	−13010.1	−26010.2
SGE	$\widehat{\mu} = 1.668 \times 10^{-2}$, $\widehat{\lambda} = 9.696 \times 10^{-1}$, $\widehat{k} = 1.364$	−13108.1	−26204.2
SST0	$\widehat{\mu} = 2.170 \times 10^{-2}$, $\widehat{\gamma} = 9.492 \times 10^{-1}$, $\widehat{\nu} = 7.461$	−13097.2	−26182.5
NIG	$\widehat{\mu} = 2.095 \times 10^{-2}$, $\widehat{\alpha} = 2.182$, $\widehat{\beta} = -8.861 \times 10^{-2}$	−13101.5	−26191.0

AIC values for models fitted to West Texas intermediate crude oil price returns. Table 5 gives parameter estimates, log-likelihood values, and AIC values for models fitted to Gold price returns. Table 6 gives parameter estimates, log-likelihood values, and AIC values for models fitted to Silver price returns. In order to avoid excessive details, the parameter estimates for the volatility component of the GARCH models are not given.

According to the AIC values in Table 2, the best fitting model for Cocoa bean price returns is the AST distribution. By comparing the likelihood values of the AST distribution ($\log L = 15{,}479.6$) and the SST distribution ($\log L = 15{,}148.2$) by the likelihood

Table 4 Fitted models and estimates for West Texas intermediate crude oil price returns

Model	Parameter estimates	$-\log L$	AIC
NORM	$\widehat{\mu} = 1.331 \times 10^{-2}$	−12545.8	−25083.5
ST	$\widehat{\mu} = 3.176 \times 10^{-2}$, $\widehat{\nu} = 7.051$	−12691.0	−25372.1
SST	$\widehat{\mu} = 1.020 \times 10^{-1}$, $\widehat{\nu} = 7.138$, $\widehat{\alpha} = 5.250 \times 10^{-1}$	−12694.9	−25377.8
AST	$\widehat{\mu} = 6.449 \times 10^{-2}$, $\widehat{\nu_1} = 6.152$, $\widehat{\nu_2} = 8.502$, $\widehat{\alpha} = 5.111 \times 10^{-1}$	−12696.0	−25377.9
GE	$\widehat{\mu} = 1.521 \times 10^{-2}$, $\widehat{k} = 1.334$	−12709.2	−25408.4
SEP	$\widehat{\mu} = 1.856 \times 10^{-2}$, $\widehat{p} = 1.313$, $\widehat{\alpha} = 5.019 \times 10^{-1}$	−12736.8	−25461.6
AEP	$\widehat{\mu} = -1.933 \times 10^{-5}$, $\widehat{p_1} = 1.076$, $\widehat{p_2} = 1.249$, $\widehat{\alpha} = 4.801 \times 10^{-1}$	−12755.1	−25496.2
SNORM	$\widehat{\mu} = 6.610 \times 10^{-3}$, $\widehat{\lambda} = 9.435 \times 10^{-1}$	−12553.0	−25096.0
SGE	$\widehat{\mu} = 8.583 \times 10^{-3}$, $\widehat{\lambda} = 1.005$, $\widehat{k} = 1.155$	−12747.0	−25482.0
SST0	$\widehat{\mu} = 1.650 \times 10^{-2}$, $\widehat{\gamma} = 9.569 \times 10^{-1}$, $\widehat{\nu} = 5.427$	−12728.2	−25444.3
NIG	$\widehat{\mu} = 1.986 \times 10^{-2}$, $\widehat{\alpha} = 2.038$, $\widehat{\beta} = -8.804 \times 10^{-2}$	−12560.0	−25108.0

ratio test, we see that the degree of freedom parameters, ν_1 and ν_2, are significantly different. The right tail of the returns is heavier. The left tail of the returns is lighter.

According to the AIC values in Table 3, the best fitting model for Brent crude oil price returns is the AEP distribution. By comparing the likelihood values of the AEP distribution ($\log L = 13{,}110.5$) and the SEP distribution ($\log L = 13{,}095.5$) by the likelihood ratio test, we see that the shape parameters, p_1 and p_2, are significantly different. The left tail of the returns is heavier. The right tail of the returns is lighter.

According to the AIC values in Table 4, the best fitting model for West Texas intermediate crude oil price returns is the AEP distribution. By comparing the likelihood values of the AEP distribution ($\log L = 12{,}755.1$) and the SEP distribution ($\log L = 12{,}736.8$) by the likelihood ratio test, we see that the shape parameters, p_1 and p_2, are significantly different. The left tail of the returns is heavier. The right tail of the returns is lighter.

According to the AIC values in Table 5, the best fitting model for Gold price returns is the AEP distribution. By comparing the likelihood values of the AEP distribution ($\log L = 17{,}787.7$) and the SEP distribution ($\log L = 17{,}785.2$) by the likelihood ratio test, we see that the shape parameters, p_1 and p_2, are significantly different. The left tail of the returns is heavier. The right tail of the returns is lighter.

According to the AIC values in Table 6, the best fitting model for Silver price returns is the SEP distribution, the particular of the AEP distribution for $p_1 = p_2$. By comparing the likelihood values of the AEP distribution ($\log L = 14{,}027.8$) and the SEP distribution ($\log L = 14{,}027.4$) by the likelihood ratio test, we see no evidence to suggest that the shape parameters, p_1 and p_2, are significantly different. So, the left and right tails of the returns behave similarly.

We see that the best fitting model for each of the data sets on stock price returns is one of the two recently introduced distributions, the AST distribution, or the AEP distribution. None of the existing or commonly used models for Z_t provides the best

Table 5 Fitted models and estimates for Gold price returns

Model	Parameter estimates	$-\log L$	AIC
NORM	$\widehat{\mu} = 3.686 \times 10^{-2}$	−17431.9	−34855.7
ST	$\widehat{\mu} = 5.885 \times 10^{-2}$, $\widehat{\nu} = 3.821$	−17745.3	−35480.6
SST	$\widehat{\mu} = 5.479 \times 10^{-2}$, $\widehat{\nu} = 3.939$, $\widehat{\alpha} = 4.990 \times 10^{-1}$	−17745.8	−35479.6
AST	$\widehat{\mu} = 5.527 \times 10^{-2}$, $\widehat{\nu_1} = 3.636$, $\widehat{\nu_2} = 3.969$, $\widehat{\alpha} = 4.986 \times 10^{-1}$	−17746.3	−35478.7
GE	$\widehat{\mu} = 1.652 \times 10^{-6}$, $\widehat{k} = 1.006$	−17781.6	−35553.2
SEP	$\widehat{\mu} = -6.267 \times 10^{-8}$, $\widehat{p} = 1.005$, $\widehat{\alpha} = 4.869 \times 10^{-1}$	−17785.2	−35558.3
AEP	$\widehat{\mu} = -9.930 \times 10^{-8}$, $\widehat{p_1} = 9.672 \times 10^{-1}$, $\widehat{p_2} = 1.042$, $\widehat{\alpha} = 4.777 \times 10^{-1}$	−17787.7	−35561.4
SNORM	$\widehat{\mu} = 3.761 \times 10^{-2}$, $\widehat{\lambda} = 1.004$	−17431.9	−34853.8
SGE	$\widehat{\mu} = 3.687 \times 10^{-2}$, $\widehat{\lambda} = 1.026$, $\widehat{k} = 1.005$	−17785.1	−35558.3
SST0	$\widehat{\mu} = 3.636 \times 10^{-2}$, $\widehat{\gamma} = 9.904 \times 10^{-1}$, $\widehat{\nu} = 3.829$	−17747.2	−35482.4
NIG	$\widehat{\mu} = 2.218 \times 10^{-2}$, $\widehat{\alpha} = 2.148$, $\widehat{\beta} = -9.497 \times 10^{-2}$	−17755.5	−35499.0

Table 6 Fitted models and estimates for Silver price returns

Model	Parameter estimates	$-\log L$	AIC
NORM	$\widehat{\mu} = 2.694 \times 10^{-2}$	−13751.4	−27494.8
ST	$\widehat{\mu} = 2.800 \times 10^{-2}$, $\widehat{\nu} = 8.413$	−13912.2	−27814.3
SST	$\widehat{\mu} = 2.745 \times 10^{-2}$, $\widehat{\nu} = 8.417$, $\widehat{\alpha} = 4.998 \times 10^{-1}$	−13912.2	
AST	$\widehat{\mu} = 4.933 \times 10^{-2}$, $\widehat{\nu_1} = 9.416$, $\widehat{\nu_2} = 7.644$, $\widehat{\alpha} = 5.083 \times 10^{-1}$	−13912.5	−27811
GE	$\widehat{\mu} = 7.075 \times 10^{-7}$, $\widehat{k} = 1.065$	−14025.5	−28041
SEP	$\widehat{\mu} = -4.391 \times 10^{-7}$, $\widehat{p} = 1.065$, $\widehat{\alpha} = 4.905 \times 10^{-1}$	−14027.4	−28042.8
AEP	$\widehat{\mu} = 5.434 \times 10^{-9}$, $\widehat{p_1} = 1.046$, $\widehat{p_2} = 1.079$, $\widehat{\alpha} = 4.866 \times 10^{-1}$	−14027.8	−28041.7
SNORM	$\widehat{\mu} = 1.664 \times 10^{-6}$, $\widehat{\lambda} = 2.746 \times 10^{-2}$	−13751.4	−27492.8
SGE	$\widehat{\mu} = 2.718 \times 10^{-2}$, $\widehat{\lambda} = 1.019$, $\widehat{k} = 1.065$	−14027.4	−28042.8
SST0	$\widehat{\mu} = 2.801 \times 10^{-2}$, $\widehat{\gamma} = 1.002$, $\widehat{\nu} = 4.372$	−14002.6	−27993.3
NIG	$\widehat{\mu} = 2.633 \times 10^{-2}$, $\widehat{\alpha} = 1.042$, $\widehat{\beta} = -4.255 \times 10^{-3}$	−14005.9	−27999.9

fits. Furthermore, for four of the five data sets on stock price returns, the tails of the returns are asymmetric. The tails are symmetric only for Silver.

The best fitting models are summarized in Table 7. Also given in this table are p values for the best fitting models based on the Cramer–von Mises statistic, the Kolmogorov–Smirnov statistic, and the Pearson chi-square statistic. These p values suggest that each best fitting model provides an adequate description of the data on price returns. The p values appear largest for Gold price returns. They appear second largest for Brent crude oil price returns. They appear smallest for Cocoa bean price returns, West Texas intermediate crude oil price returns, and Silver price returns.

We now give some measures of goodness of the best fitted models. These measures are obtained by comparing the observed values of mean, standard deviation, and value

Table 7 Best fitting models

	Cocoa bean	Brent crude oil	West Texas intermediate crude oil	Gold	Silver
Best model	AST	AEP	AEP	AEP	SEP
CVM test p value	0.061	0.094	0.064	0.134	0.060
KS test p value	0.059	0.089	0.061	0.223	0.051
Pearson test p value	0.052	0.088	0.066	0.185	0.063

Table 8 Mean absolution deviations as measures of goodness of the best fitting models

	Cocoa bean	Brent crude oil	West Texas intermediate crude oil	Gold	Silver
Mean ($w = 10$)	4.938×10^{-3}	5.301×10^{-3}	5.453×10^{-3}	2.263×10^{-3}	4.192×10^{-3}
SD ($w = 10$)	–	2.148×10^{-4}	2.998×10^{-4}	4.842×10^{-5}	2.249×10^{-4}
$\mathrm{VaR}_{0.9}$ ($w = 10$)	1.107×10^{-2}	8.582×10^{-3}	9.510×10^{-3}	4.030×10^{-3}	8.240×10^{-3}
$\mathrm{VaR}_{0.99}$ ($w = 10$)	6.960×10^{-2}	2.221×10^{-2}	2.757×10^{-2}	1.323×10^{-2}	2.659×10^{-2}
Mean ($w = 50$)	3.578×10^{-3}	2.350×10^{-3}	2.264×10^{-3}	9.679×10^{-4}	1.758×10^{-3}
SD ($w = 50$)	–	1.168×10^{-4}	1.558×10^{-4}	2.664×10^{-5}	1.520×10^{-4}
$\mathrm{VaR}_{0.9}$ ($w = 50$)	9.125×10^{-3}	4.239×10^{-3}	4.636×10^{-3}	2.162×10^{-3}	4.804×10^{-3}
$\mathrm{VaR}_{0.99}$ ($w = 50$)	5.701×10^{-2}	1.011×10^{-2}	1.436×10^{-2}	7.968×10^{-3}	1.647×10^{-2}
Mean ($w = 100$)	3.450×10^{-3}	1.566×10^{-3}	1.620×10^{-3}	7.047×10^{-4}	1.156×10^{-3}
SD ($w = 100$)	–	1.276×10^{-4}	1.408×10^{-4}	3.206×10^{-5}	1.465×10^{-4}
$\mathrm{VaR}_{0.9}$ ($w = 100$)	8.968×10^{-3}	4.072×10^{-3}	4.309×10^{-3}	1.993×10^{-3}	4.397×10^{-3}
$\mathrm{VaR}_{0.99}$ ($w = 100$)	5.439×10^{-2}	9.738×10^{-3}	1.278×10^{-2}	7.253×10^{-3}	1.374×10^{-2}

at risk over windows of length w with fitted values. We use two criteria for comparison: mean absolute deviation and root mean squared error. Table 8 gives the mean absolute deviations for mean, standard deviation, $\mathrm{VaR}_{0.9}$, and $\mathrm{VaR}_{0.99}$ for $w = 10, 50, 100$ days. Table 9 gives the root mean squared errors for mean, standard deviation, $\mathrm{VaR}_{0.9}$, and $\mathrm{VaR}_{0.99}$ for $w = 10, 50, 100$ days. The standard deviation for Cocoa bean does not exist since its best fitting model is the AST distribution with $\widehat{\nu_2} = 1.827 < 2$.

The mean absolute deviations and the root mean squared errors appear small enough to suggest that the best fitting models are reasonable. The mean absolute deviations and the root mean squared errors appear smallest for Gold price returns. They appear largest for Cocoa bean price returns, West Texas intermediate crude oil price returns, and Silver price returns. However, there is no evidence to suggest that the mean absolute deviations or the root mean squared errors vary significantly with respect to w.

Boxplots of the fitted values of VaR_p, $p = 0.9, 0.95, 0.975, 0.99$ for the five commodities are shown in Fig. 2. We can observe the following: the median of value at risk is largest for West Texas intermediate crude oil and smallest for Gold when $p = 0.9$ or $p = 0.95$; the median of value at risk is largest for Cocoa bean and smallest

Table 9 Root mean squared errors as measures of goodness of the best fitting models

	Cocoa bean	Brent crude oil	West Texas intermediate crude oil	Gold	Silver
Mean ($w = 10$)	6.657×10^{-3}	6.899×10^{-3}	7.192×10^{-3}	3.038×10^{-3}	5.825×10^{-3}
SD ($w = 10$)	–	3.345×10^{-4}	5.394×10^{-4}	9.311×10^{-5}	5.927×10^{-4}
$\mathrm{VaR}_{0.9}$ ($w = 10$)	1.647×10^{-2}	1.130×10^{-2}	1.254×10^{-2}	5.438×10^{-3}	1.144×10^{-2}
$\mathrm{VaR}_{0.99}$ ($w = 10$)	8.623×10^{-2}	2.593×10^{-2}	3.153×10^{-2}	1.567×10^{-2}	3.164×10^{-2}
Mean ($w = 50$)	4.502×10^{-3}	3.226×10^{-3}	3.133×10^{-3}	1.246×10^{-3}	2.319×10^{-3}
SD ($w = 50$)	–	1.768×10^{-4}	2.602×10^{-4}	4.718×10^{-5}	3.323×10^{-4}
$\mathrm{VaR}_{0.9}$ ($w = 50$)	6.580×10^{-2}	5.839×10^{-3}	6.209×10^{-3}	2.949×10^{-3}	7.108×10^{-3}
$\mathrm{VaR}_{0.99}$ ($w = 50$)	1.005×10^{-1}	1.254×10^{-2}	1.798×10^{-2}	1.005×10^{-2}	2.110×10^{-2}
Mean ($w = 100$)	4.330×10^{-3}	2.399×10^{-3}	2.344×10^{-3}	8.811×10^{-4}	1.696×10^{-3}
SD ($w = 100$)	–	2.053×10^{-4}	2.534×10^{-4}	5.818×10^{-5}	2.873×10^{-4}
$\mathrm{VaR}_{0.9}$ ($w = 100$)	1.493×10^{-2}	5.793×10^{-3}	5.819×10^{-3}	2.838×10^{-3}	6.719×10^{-3}
$\mathrm{VaR}_{0.99}$ ($w = 100$)	7.355×10^{-2}	1.216×10^{-2}	1.637×10^{-2}	9.358×10^{-3}	1.806×10^{-2}

for Gold when $p = 0.975$ or $p = 0.99$; the variability of value at risk is largest for Cocoa bean and smallest for Gold for every p; and the variability of value at risk decreases with p for each commodity.

Boxplots of the fitted values of $\mathrm{ES}_p,\ p = 0.9, 0.95, 0.975, 0.99$ for the five commodities are shown in Fig. 3. We can observe the following: the median of expected shortfall is largest for Cocoa bean and smallest for Gold for every p; the variability of expected shortfall is largest for Cocoa bean and smallest for Gold for every p; and the variability of expected shortfall decreases with p for each commodity.

Figure 4 shows how the estimates of the expected volatility, $\widehat{\omega} + \widehat{\alpha_1} \widehat{E} \left(X_{i-1}^2 \right) + \widehat{\beta_1} \sigma_{i-1}^2$, vary with respect to time for the best fitting models. We can observe the following: the expected volatility for Brent crude oil and Gold increases monotonically and sharply with respect to time; the expected volatility for West Texas intermediate crude oil and Silver increases monotonically before approaching an asymptote; the expected volatility for all t is largest for Brent crude oil; the expected volatility for small t is second largest for West Texas intermediate crude oil; the expected volatility for all sufficiently large t is second largest for Gold; the expected volatility for small t is third largest for Silver; the expected volatility for all sufficiently large t is third largest for West Texas intermediate crude oil; the expected volatility for small t is smallest for Gold; and the expected volatility for all sufficiently large t is smallest for Silver. The expected volatility for Cocoa bean does not exist since its best fitting model is the AST distribution with $\widehat{\nu_2} = 1.827 < 2$.

Finally, Fig. 5 gives forecasts of $\mathrm{VaR}_p,\ p = 0.9, 0.95, 0.975, 0.99$ by one hundred additional days. We can observe the following: the forecasts for each commodity increase monotonically with respect to time; the forecasts for each commodity increase monotonically with respect to p; the forecasts are largest for Silver for every p; the forecasts are second largest for West Texas intermediate crude oil for every p; the

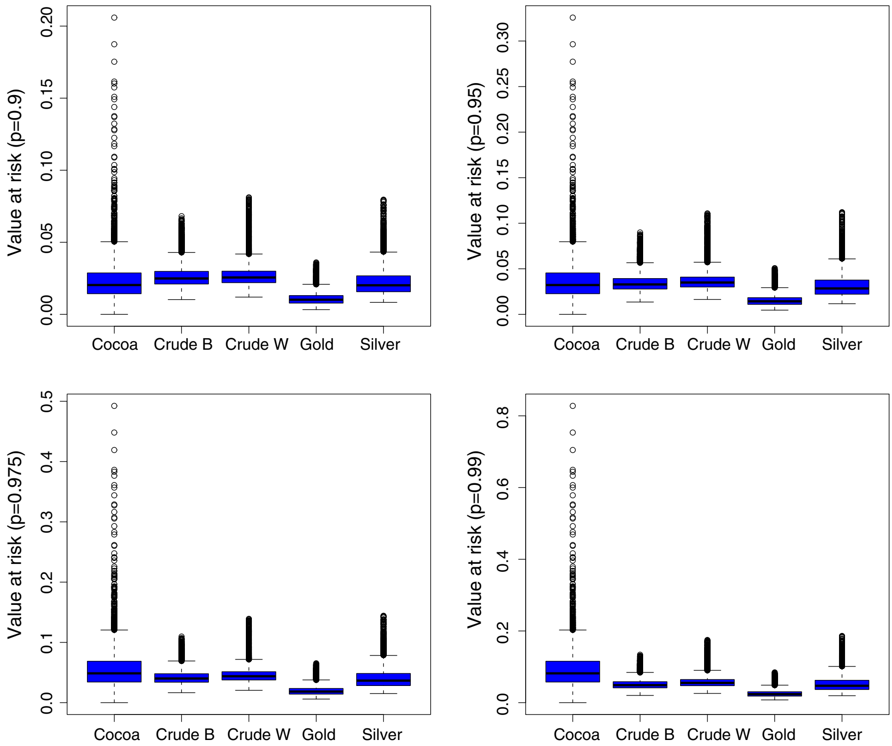

Fig. 2 Boxplots of $\text{VaR}_{0.9}$, $\text{VaR}_{0.95}$, $\text{VaR}_{0.975}$ and $\text{VaR}_{0.99}$ for Cocoa bean, Brent crude oil, West Texas intermediate crude oil, Gold, and Silver

forecasts are third largest for Brent crude oil for every p; and the forecasts are smallest for Gold for every p.

5 Relationship to other work

There is a large amount of work on modeling of the five popular commodities. A variety of modeling approaches have been used. Some recent works for modeling of gold prices have used adaptive network fuzzy inference systems (Yazdani-Chamzini et al. 2012), artificial neural networks (Yildirim et al. 2011), multifractal detrended fluctuation analysis (Bolgorian and Gharli 2011), and random walk models (Nakamura and Small 2007). Here, we compare the results reported in Sect. 4 with some of the known results and known facts.

Assis et al. (2010) find a positive linear trend in the price of cocoa beans, consistent with our Fig. 5. Idris et al. (2011) identify the decrease in world production of cocoa beans in 2007 and the stagnant trend in the demand as factors contributing to the rise in cocoa beans price, see Fig. 5. Maurice and Davis (2011) argue that there is a long-run equilibrium relationship between oil prices and cocoa prices, meaning that changes in oil prices directly affect cocoa prices. This is consistent with Figs. 4 and 5.

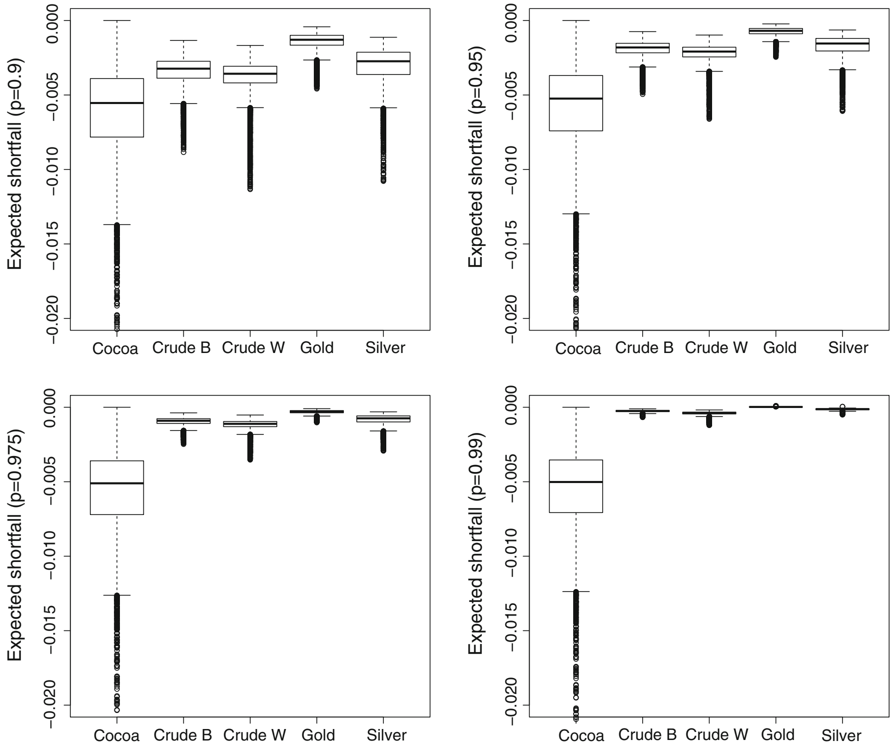

Fig. 3 Boxplots of $ES_{0.9}$, $ES_{0.95}$, $ES_{0.975}$ and $ES_{0.99}$ for Cocoa bean, Brent crude oil, West Texas intermediate crude oil, Gold, and Silver

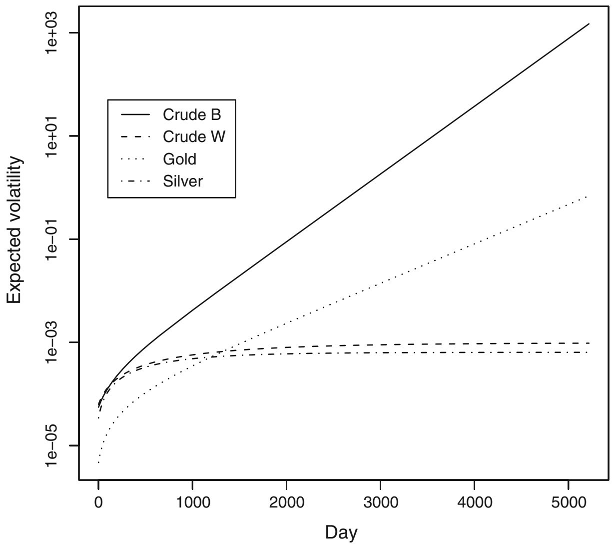

Fig. 4 Expected volatility versus time for Brent crude oil, West Texas intermediate crude oil, Gold, and Silver. The y axis is in log scale

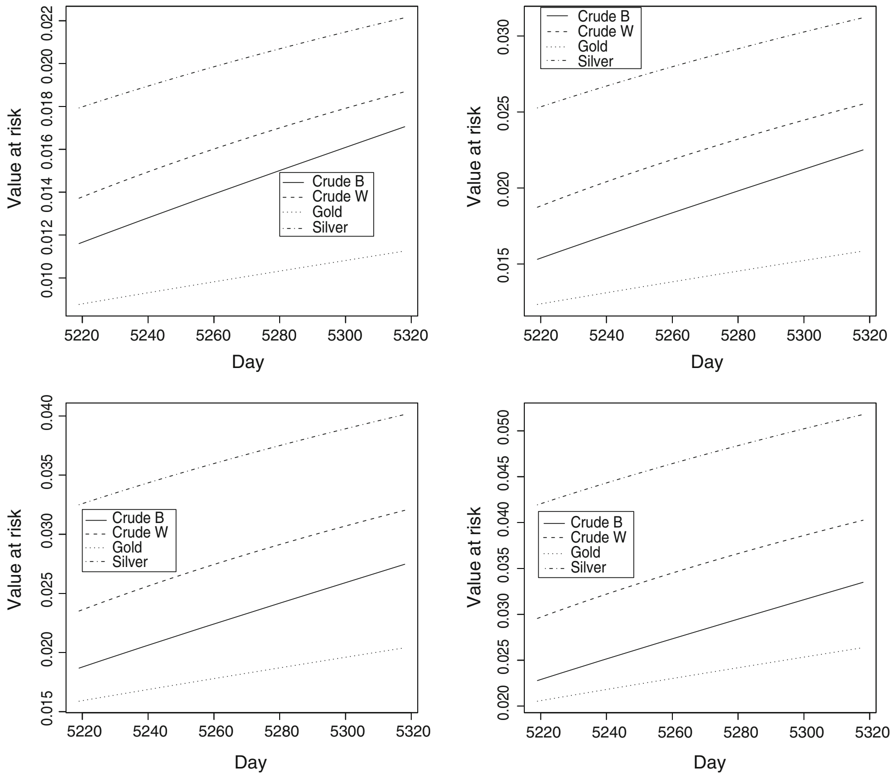

Fig. 5 Forecasts of $\text{VaR}_{0.9}$, $\text{VaR}_{0.95}$, $\text{VaR}_{0.975}$ and $\text{VaR}_{0.99}$ for Brent crude oil, West Texas intermediate crude oil, Gold, and Silver

Ubilava and Helmers (2011) argue that the price rise may be due to El Niño Southern Oscillation (ENSO), a climatic anomaly affecting temperature, and precipitation. The regions growing cocoa beans are most affected by ENSO.

Zhang and Wei (2010) observe that "the crude oil price volatility magnitude proves greater than that of the gold price in the sampling period. From the value of coefficient of variance, it can be found that the fluctuation of crude oil price is 2 times stronger than that of gold price." This is consistent with our results in Table 1 and Fig. 4.

Cheong (2011) identifies OPEC's inability to control the prices using the supply and demand scheme, international tension among petroleum produces, and energy crises as factors contributing to the rise in oil prices and their volatility, see Figs. 4 and 5. Novotny (2012) identifies industrial production in OECD countries and short-term real interest rates in the USA as factors contributing to the increase in oil price, see Fig. 5. Novotny (2012) states "that since 2005 a depreciation of the nominal effective exchange rate of the dollar of 1 % has implied an increase in the oil price of 2.1 %."

Shafiee and Topal (2010) predict that "gold price would stay abnormally high up to the end of 2014. After that, the price would revert to the long-term trend until 2018." This is consistent with our forecasts in Fig. 5. Lu (2011) identifies the devaluation of the US dollar, inflation of prices from all over the world, and the US sub-prime mortgage crisis as factors contributing to the rise in gold price, see Fig. 5. Deepika et al. (2012) identify the world stock prices, US dollar index, and inflation as factors contributing to the rise in gold price. According to Erb and Harvey (2012), the real price of gold has been on the increase for at least twenty-three countries, including the United States. Lee et al. (2012) identify a unidirectional relationship between West Texas intermediate crude oil and gold as a factor affecting the price of the latter. According to the World Gold Council (2012), booming Chinese and Indian economies (India and China together consume annually, more than 60 % of the total gold produced) are driving the rise in gold price. Ziaei (2012) identifies the recent worldwide financial crisis and instability such as recession and deficit problems in the Euro zone and the US as factors contributing to the rise in gold price. Reboredo (2013) identifies oil price rise, inflation, oil-exporting countries having gold in their international reserve portfolios, US dollar exchange rate, and US dollar depreciation as factors contributing to the rise in gold price.

James (2010) says that the price of silver is set to rise with the price of gold, which is consistent as shown in Fig. 5.

Agnolucci (2009) fitted GARCH models based on the NORM, ST, and GE distributions in Sect. 3 to West Texas intermediate crude oil price returns. He found that the GE distribution gave the best fit. This is consistent with our results in Table 4. Cheng and Hung (2011) fitted GARCH models based on the NORM, GE, and SGE distributions in Sect. 3 to stock price returns on six commodities, West Texas intermediate crude oil, gasoline, heating oil, gold, silver, and copper. They found that the SGE distribution gave the best fit for all commodities. This is consistent with our results in Tables 4, 5, and 6. Also one of the conclusions "the mean VaR estimates of petroleum commodities are relatively higher than those of the metal commodities" in Cheng and Hung (2011) is consistent with our Figs. 2 and 5.

But none of the known results has identified asymmetric tails in the distribution of the returns of the commodities. Our results in Sect. 4 appear to be the first results identifying asymmetric tails in the distribution of the returns of Cocoa beans price, Brent crude oil price, West Texas intermediate crude oil price, and Gold price. We did not identify asymmetric tails for the Silver price returns.

6 Conclusions

We have provided GARCH modeling of five popular commodities: Cocoa bean, Brent crude oil, West Texas intermediate crude oil, Gold, and Silver. For each commodity, the GARCH(1, 1) model was fitted with the following innovation distributions: the NORM distribution, the SNORM distribution, the ST distribution, the SST0 distribution, the GE distribution, the SGE distribution, the NIG distribution, the AEP distribution, the SEP distribution, the AST distribution, and the SST distribution. In each case, one of the four last distributions was shown to give the best fit. The following descriptions are given for the best fitting models: (i) measure of goodness of fit based on mean absolute deviation and root mean squared error; (ii) comparison of value at risk and expected shortfall among the five commodities; (iii) comparison of expected volatility among the five commodities; and (iv) comparison of forecasts of value at risk among the five commodities. We have also compared these descriptions with published results and known facts.

The work in this paper can be extended in several ways. One is to study the dependence between two or more of the commodities (for example, the dependence between oil prices and gold prices), and another is to include time as a covariate in the modeling.

Finally, we comment on the AEP and AST distributions, the two best fitting models. Zhu and Galbraith (2011) state that the AEP and AST distributions "control skewness and the thickness of each tail, have greater flexibility to use information in a large sample of data, and avoid constraining the left and right tails to have the same thickness." In doing so, they "can potentially obtain better estimates of the thickness of the left tail, with corresponding potential improvements in forecasting power for risk of loss" (Zhu and Galbraith 2011). None of the known distributions treat the tails so differently in the way the AEP and AST distributions do. Hence, the AEP and AST distributions can be expected to provide better fits and better forecasts whenever there are asymmetric tails in the data.

Asymmetric tails are common in finance. Some reasons for expecting asymmetric tails are response of spot prices to shocks in one-month futures oil prices is much steeper in high spot prices than in low spot prices (Lee and Zeng 2011); and bad news in the oil market has the potentiality of increasing volatility in the oil price than good news (Salisu and Fasanya 2013). Some recent applications involving asymmetric tails and in particular asymmetric heavy tails are tails of the profit distributions of Hollywood movies (Vany and Walls 2002); tails in energy markets volatility (Aloui and Mabrouk 2010); and tails of a momentum strategy's return distribution (Gregory-Allen et al. 2012).

Acknowledgments The authors would like to thank the Editor and the two referees for careful reading and for their comments which greatly improved the paper.

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