MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
QUIZ PROBLEM 9

Among N independent random variable having the Binomial $(2, p)$ distribution, n_0 take on the value 0, n_1 take on the value 1 while n_2 take on the value 2. We have $n_0 + n_1 + n_2 = N$.

Let X denote a random variable having the Binomial $(2, p)$ distribution. The likelihood function of p is

$$\begin{aligned} L(p) &= [\Pr(X=0)]^{n_0} [\Pr(X=1)]^{n_1} [\Pr(X=2)]^{n_2} \\ &= \left[\binom{2}{0} p^0 (1-p)^2\right]^{n_0} \left[\binom{2}{1} p^1 (1-p)^1\right]^{n_1} \left[\binom{2}{2} p^2 (1-p)^0\right]^{n_2} \\ &= \left[(1-p)^2\right]^{n_0} [2p(1-p)]^{n_1} \left[p^2\right]^{n_2} \end{aligned}$$

and so the log–likelihood function is

$$l(p) = n_1 \log 2 + (2n_0 + n_1) \log(1-p) + (n_1 + 2n_2) \log p.$$

The first derivative of $l(p)$ is

$$\frac{dl(p)}{dp} = \frac{n_1 + 2n_2}{p} - \frac{2n_0 + n_1}{1-p}$$

and setting this to zero gives the solution $\hat{p} = (n_1 + 2n_2)/2N$. This is indeed the mle since the second derivative

$$\frac{d^2 l(p)}{dp^2} = -\frac{n_1 + 2n_2}{p^2} - \frac{2n_0 + n_1}{(1-p)^2} < 0.$$