MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTION TO QUIZ PROBLEM 4

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability mass function

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 1, 2, \ldots, n$ and $0 < p < 1$ is an unknown parameter. The likelihood function of p is

$$L(p) = \prod_{i=1}^{n} \left[\binom{n}{X_i} p^{X_i} (1-p)^{n-X_i} \right] = \prod_{i=1}^{n} \binom{n}{X_i} p^{\sum_{i=1}^{n} X_i} (1-p)^{n^2 - \sum_{i=1}^{n} X_i}.$$

The log likelihood function is

$$\log L(p) = \sum_{i=1}^{n} \log \binom{n}{X_i} + \left(\sum_{i=1}^{n} X_i \right) \log p + \left(n^2 - \sum_{i=1}^{n} X_i \right) \log(1-p)$$

The first derivative is

$$\frac{d \log L(p)}{dp} = \left(\sum_{i=1}^{n} X_i \right) \frac{1}{p} - \left(n^2 - \sum_{i=1}^{n} X_i \right) \frac{1}{1-p}.$$

Setting this to zero, we see

$$\widehat{p} = \frac{1}{n^2} \sum_{i=1}^{n} X_i = \frac{\overline{X}}{n}.$$

where $\overline{X}$ denotes the sample mean. This is an MLE since

$$\frac{d^2 \log L(p)}{dp^2} = - \left(\sum_{i=1}^{n} X_i \right) \frac{1}{p^2} - \left(n^2 - \sum_{i=1}^{n} X_i \right) \frac{1}{(1-p)^2} < 0.$$