MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTION TO QUIZ PROBLEM 7

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability mass function

$$p(x) = \frac{(\lambda x)^{x-1} \exp(-\lambda x)}{x!}$$

for $x = 1, 2, \ldots$ and $0 < \lambda < 1$ is an unknown parameter. The likelihood function of λ is

$$L(\lambda) = \prod_{i=1}^{n} \frac{(\lambda X_i)^{X_i-1} \exp(-\lambda X_i)}{X_i!} = \frac{\prod_{i=1}^{n} (\lambda X_i)^{X_i-1} \exp\left(-\lambda \sum_{i=1}^{n} X_i\right)}{\prod_{i=1}^{n} X_i!}.$$

The log likelihood function is

$$\log L(\lambda) = \sum_{i=1}^{n} (X_i - 1) \log(\lambda X_i) - \lambda \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \log X_i!.$$

The first derivative is

$$\frac{d \log L(\lambda)}{d\lambda} = \frac{1}{\lambda} \sum_{i=1}^{n} (X_i - 1) - \sum_{i=1}^{n} X_i.$$

Setting this to zero, we see

$$\widehat{\lambda} = \frac{\sum_{i=1}^{n} (X_i - 1)}{\sum_{i=1}^{n} X_i} = \frac{\overline{X} - 1}{\overline{X}},$$

where $\overline{X}$ denotes the sample mean. This is an MLE since

$$\frac{d^2 \log L(\lambda)}{d\lambda^2} = -\frac{1}{\lambda^2} \sum_{i=1}^{n} (X_i - 1) < 0,$$

excluding the case that $X_i = 1$ for all i.