MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTION TO QUIZ PROBLEM 6

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $\mathrm{Uni}[0, \theta]$. Let $Z = \max(X_1, X_2, \ldots, X_n)$. The cdf of Z is

$$
\begin{aligned}
F_Z(z) &= \Pr\left[\max(X_1, X_2, \ldots, X_n) \leq z\right] \\
&= \Pr\left[X_1 \leq z, X_2 \leq z, \ldots, X_n \leq z\right] \\
&= \Pr\left[X_1 \leq z\right] \Pr\left[X_2 \leq z\right] \cdots \Pr\left[X_n \leq z\right] \\
&= \frac{z}{\theta}\frac{z}{\theta} \cdots \frac{z}{\theta} \\
&= \frac{z^n}{\theta^n}.
\end{aligned}
$$

So the pdf of Z is

$$
f_Z(z) = \frac{nz^{n-1}}{\theta^n}
$$

for $0 < z < \theta$. Hence, the bias of Z as an estimator of θ is

$$
\begin{aligned}
\mathrm{Bias}(Z) &= E(Z) - \theta \\
&= \int_0^\theta z \frac{nz^{n-1}}{\theta^n} dz - \theta \\
&= \frac{n}{\theta^n} \int_0^\theta z^n dz - \theta \\
&= \frac{n}{\theta^n} \left[\frac{z^{n+1}}{n+1}\right]_0^\theta - \theta \\
&= \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} - \theta \\
&= \frac{n\theta}{n+1} - \theta \\
&= -\frac{\theta}{n+1}.
\end{aligned}
$$

The variance of Z is

$$
\begin{aligned}
\mathrm{Var}(Z) &= E\left(Z^2\right) - \left[\frac{n\theta}{n+1}\right]^2 \\
&= \int_0^\theta z^2 \frac{nz^{n-1}}{\theta^n} dz - \left[\frac{n\theta}{n+1}\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{n}{\theta^n}\int_0^\theta z^{n+1}dz - \left[\frac{n\theta}{n+1}\right]^2 \\
&= \frac{n}{\theta^n}\left[\frac{z^{n+2}}{n+2}\right]_0^\theta - \left[\frac{n\theta}{n+1}\right]^2 \\
&= \frac{n}{\theta^n}\frac{\theta^{n+2}}{n+2} - \left[\frac{n\theta}{n+1}\right]^2 \\
&= \frac{n\theta^2}{n+2} - \left[\frac{n\theta}{n+1}\right]^2 \\
&= n\theta^2\left[\frac{1}{n+2} - \frac{n}{(n+1)^2}\right] \\
&= n\theta^2\frac{(n+1)^2 - n^2 - 2n}{(n+1)^2(n+2)} \\
&= \theta^2\frac{n}{(n+1)^2(n+2)}.
\end{aligned}
$$

The MSE of Z is

$$
\mathrm{MSE}(Z) = \theta^2\frac{n}{(n+1)^2(n+2)} + \left[\frac{\theta}{n+1}\right]^2
$$

which approaches zero as $n \to \infty$. Hence, Z is consistent for θ.