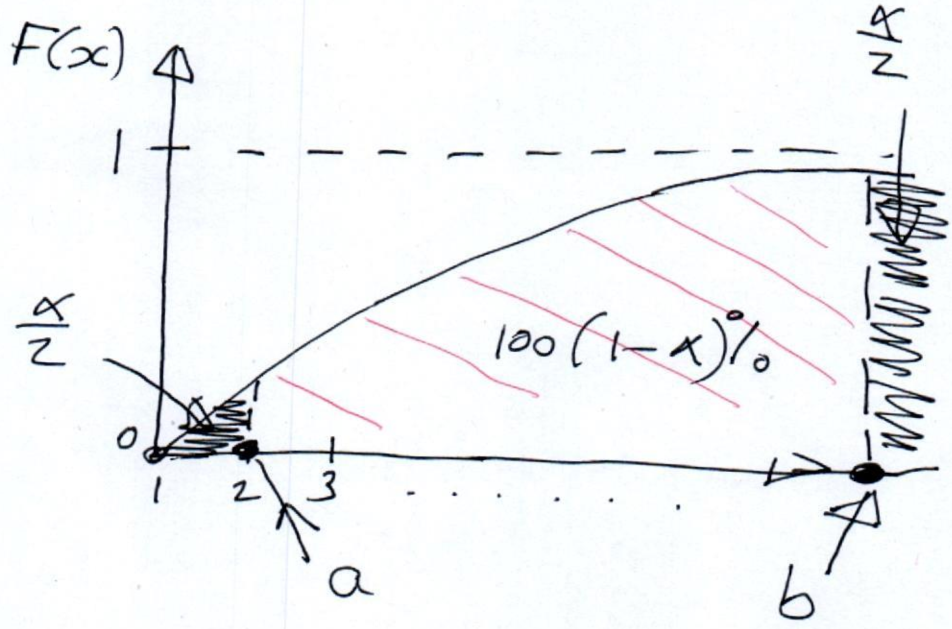
Example 5

Suppose $X \sim Geom(p)$.

Find a $100(1-\alpha)\%$ CI for p.

PMF $P(x) = p(1-p)^{x-1}, \quad x = 1, 2, \ldots$

CDF $F(x) = 1 - (1-p)^x, \quad x = 1, 2, \ldots$



$F(a) = \frac{\alpha}{2}$

$\Rightarrow 1 - (1-p)^a = \frac{\alpha}{2}$

$\Rightarrow (1-p)^a = 1 - \frac{\alpha}{2}$

$\Rightarrow a \log(1-p) = \log(1 - \frac{\alpha}{2})$

$$\Rightarrow \quad a = \frac{\log(1-\frac{\alpha}{2})}{\log(1-p)}$$

$$F(b) = 1 - \frac{\alpha}{2}$$

$$\Rightarrow 1 - (1-p)^b = 1 - \frac{\alpha}{2}$$

$$\Rightarrow (1-p)^b = \frac{\alpha}{2}$$

$$\Rightarrow b \log(1-p) = \log\left(\frac{\alpha}{2}\right)$$

$$\Rightarrow b = \frac{\log\left(\frac{\alpha}{2}\right)}{\log(1-p)}$$

$$\Rightarrow P\left(\frac{\log\left(1-\frac{\alpha}{2}\right)}{\log(1-p)} < X < \frac{\log\left(\frac{\alpha}{2}\right)}{\log(1-p)}\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{\log\left(\frac{\alpha}{2}\right)}{X} < \log(1-p) < \frac{\log\left(1-\frac{\alpha}{2}\right)}{X}\right) = 1-\alpha$$

$$\Rightarrow P\left(\exp\left[\frac{\log\left(\frac{\alpha}{2}\right)}{X}\right] < 1-p < \exp\left[\frac{\log\left(1-\frac{\alpha}{2}\right)}{X}\right]\right) = 1-\alpha$$

$$\Rightarrow P\left(1-\exp\left[\frac{\log\left(1-\frac{\alpha}{2}\right)}{X}\right] < p < 1-\exp\left[\frac{\log\left(\frac{\alpha}{2}\right)}{X}\right]\right) = 1-\alpha$$

Hence, a $100(1-\alpha)\%$ CI for p is

$$\left(1-\exp\left[\frac{\log\left(1-\frac{\alpha}{2}\right)}{X}\right], \; 1-\exp\left[\frac{\log\left(\frac{\alpha}{2}\right)}{X}\right]\right).$$

Example 6 Suppose $X_1, \ldots, X_n$ are IID $LN(0, \theta^2)$. Find a $100(1-\alpha)\%$ CI for θ.

$$X_i \sim LN(\underline{\underline{0}}, \theta^2)$$

$$\Rightarrow \log X_i \sim N(\underline{\underline{0}}, \theta^2)$$

$$\Rightarrow \sum_{i=1}^{n} \left(\frac{\log X_i - 0}{\theta} \right)^2 \sim \chi^2(n) \quad \text{by prop 6.}$$

$$\Rightarrow \sum_{i=1}^{n} \left(\frac{\log X_i}{\theta} \right)^2 \sim \chi^2(n)$$

$$\Rightarrow \frac{1}{\theta^2} \sum_{i=1}^{n} (\log X_i)^2 \sim \chi^2(n).$$

$$\Rightarrow P\left(\chi^2_{n, \frac{\alpha}{2}} < \frac{1}{\theta^2} \sum_{i=1}^{n} (\log X_i)^2 < \chi^2_{n, 1-\frac{\alpha}{2}} \right) = 1 - \alpha$$

$$\Rightarrow P\left(\frac{\sum_{i=1}^{n} (\log X_i)^2}{\chi^2_{n, 1-\frac{\alpha}{2}}} < \theta^2 < \frac{\sum_{i=1}^{n} (\log X_i)^2}{\chi^2_{n, \frac{\alpha}{2}}} \right) = 1 - \alpha$$

$$\Rightarrow P\left(\sqrt{\frac{\sum_{i=1}^{n}(\log X_i)^2}{\chi^2_{n,1-\frac{\alpha}{2}}}} < \theta < \sqrt{\frac{\sum_{i=1}^{n}(\log X_i)^2}{\chi^2_{n,\frac{\alpha}{2}}}}\right)$$

$$= 1 - \alpha$$

Hence, a $100(1-\alpha)\%$ CI for θ is

$$\left(\sqrt{\frac{\sum_{i=1}^{n}(\log X_i)^2}{\chi^2_{n,1-\frac{\alpha}{2}}}}, \sqrt{\frac{\sum_{i=1}^{n}(\log X_i)^2}{\chi^2_{n,\frac{\alpha}{2}}}}\right).$$

Example 7 Suppose $X \sim N(0, a^2)$ and $Y \sim N(0, b^2)$ are intepentent. Find a $100(1-\alpha)\%$ CI for $\sqrt{a^2+b^2}$.

$$X \sim N(0, a^2), \quad Y \sim N(0, b^2)$$

$$\Rightarrow X+Y \sim N(0, a^2+b^2)$$

$$\Rightarrow \frac{X+Y}{\sqrt{a^2+b^2}} \sim N(0, 1)$$

$$\Rightarrow P\left(Z_{\frac{\alpha}{2}} < \frac{X+Y}{\sqrt{a^2+b^2}} < Z_{1-\frac{\alpha}{2}} \right) = 1-\alpha$$

$$\Rightarrow P\left(\sqrt{a^2+b^2}\,\boxed{Z_{\frac{\alpha}{2}}} < X+Y < \sqrt{a^2+b^2}\,\boxed{Z_{1-\frac{\alpha}{2}}} \right) = 1-\alpha$$

$$\Rightarrow P\left(\sqrt{a^2+b^2} > \frac{X+Y}{Z_{\frac{\alpha}{2}}} \ \& \ \sqrt{a^2+b^2} > \frac{X+Y}{Z_{1-\frac{\alpha}{2}}} \right) = 1-\alpha$$

$$\Rightarrow P\left(\sqrt{a^2+b^2} > \max\left(\frac{X+Y}{Z_{\frac{\alpha}{2}}}, \frac{X+Y}{Z_{1-\frac{\alpha}{2}}} \right) \right) = 1-\alpha$$

Hence, a $100(1-\alpha)\%$ CI for $\sqrt{a^2+b^2}$ is

$$\left(\max\left(\frac{X+Y}{Z_{\frac{\alpha}{2}}}, \frac{X+Y}{Z_{1-\frac{\alpha}{2}}} \right), \ \infty \right).$$

Example 8 Suppose $X \sim N(a, 1)$ and $Y \sim N(b, 1)$ are independent. Find a $100(1-\alpha)\%$ CI for $a+b$.