Example 1 Suppose $X_1, \dots, X_n$ are IID $LN(\theta, 1)$. Find a $100(1-\alpha)\%$ CI for θ.

$$X_i \sim LN(\theta, 1)$$

$$\Rightarrow \log X_i \sim N(\theta, 1)$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} \log X_i \sim N\left(\theta, \frac{1}{n}\right)$$

$$\Rightarrow \left(\frac{1}{n}\sum_{i=1}^{n} \log X_i\right) - \theta \sim N\left(0, \frac{1}{n}\right)$$

$$\Rightarrow \frac{\left(\frac{1}{n}\sum_{i=1}^{n} \log X_i\right) - \theta}{\sqrt{\frac{1}{n}}} \sim N(0, 1)$$

$$\Rightarrow \sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n} \log X_i\right) - \theta\right] \sim N(0, 1)$$

$$\Rightarrow P\left(z_{\frac{\alpha}{2}} < \sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n} \log X_i\right) - \theta\right] < z_{1-\frac{\alpha}{2}}\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{z_{\frac{\alpha}{2}}}{\sqrt{n}} < \left(\frac{1}{n}\sum_{i=1}^{n} \log X_i\right) - \theta < \frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}}\right) = 1-\alpha$$

$$\Rightarrow P\left(\left(\frac{1}{n}\sum_{i=1}^{n}\log X_i\right)-\frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}} < \theta < \left(\frac{1}{n}\sum_{i=1}^{n}\log X_i\right)-\frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right)$$

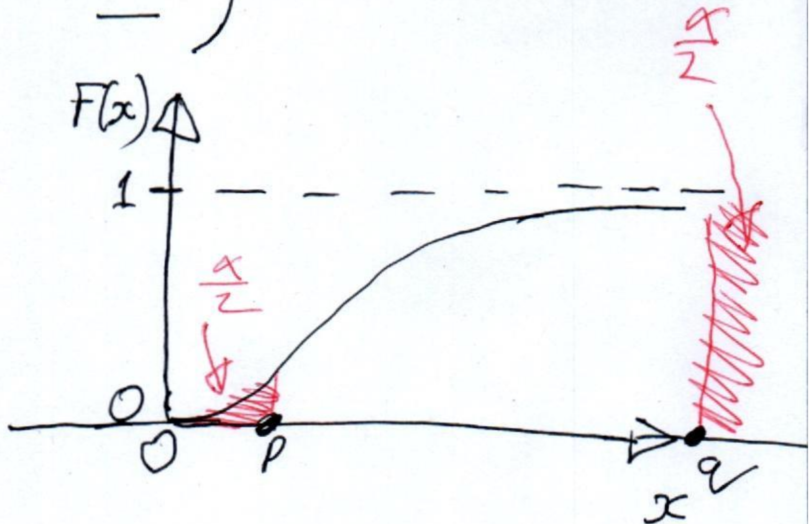
$$= 1-\alpha$$

$\Rightarrow$ Hence, a $100(1-\alpha)\%$ CI for θ is

$$\left[\left(\frac{1}{n}\sum_{i=1}^{n}\log X_i\right)-\frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}}, \left(\frac{1}{n}\sum_{i=1}^{n}\log X_i\right)-\frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right].$$

Example 2 Suppose X has the CDF $F(x) = [1-e^{-x}]^b$, $x>0$. Find a $100(1-\alpha)\%$ CI for b.

$$P(\underline{p} < X < \underline{q}) = 1-\alpha$$



$$P(X \le p) = \frac{\alpha}{2}$$

$$P(X \ge q) = \frac{\alpha}{2}$$

$$F(p) = \frac{\alpha}{2}$$

$$\Rightarrow [1-e^{-p}]^b = \frac{\alpha}{2}$$

$$\Rightarrow 1-e^{-p} = \left(\frac{\alpha}{2}\right)^{\frac{1}{b}}$$

$$\Rightarrow e^{-p} = 1-\left(\frac{\alpha}{2}\right)^{\frac{1}{b}}$$

$$\Rightarrow \boxed{p = -\log\left[1-\left(\frac{\alpha}{2}\right)^{\frac{1}{b}}\right]}$$

$$\Rightarrow \quad 1 - F(q) = \frac{\alpha}{2}$$

$$\Rightarrow \quad F(q) = 1 - \frac{\alpha}{2}$$

$$\Rightarrow \quad \boxed{q = -\log\left[1 - \left(1 - \frac{\alpha}{2}\right)^{\frac{1}{b}}\right]}$$

$$P\left(-\log\left[1 - \left(\frac{\alpha}{2}\right)^{\frac{1}{b}}\right] < X < -\log\left[1 - \left(1 - \frac{\alpha}{2}\right)^{\frac{1}{b}}\right]\right) = 1 - \alpha$$

$$\Rightarrow P\left(\left[1 - \left(\frac{\alpha}{2}\right)^{\frac{1}{b}}\right]^{-1} < e^{X} < \left[1 - \left(1 - \frac{\alpha}{2}\right)^{\frac{1}{b}}\right]^{-1}\right) = 1 - \alpha$$

$$\Rightarrow P\left(1 - \left(1 - \frac{\alpha}{2}\right)^{\frac{1}{b}} < e^{-X} < 1 - \left(\frac{\alpha}{2}\right)^{\frac{1}{b}}\right) = 1 - \alpha$$

$$\Rightarrow P\left(\left(\frac{\alpha}{2}\right)^{\frac{1}{b}} < 1 - e^{-X} < \left(1 - \frac{\alpha}{2}\right)^{\frac{1}{b}}\right) = 1 - \alpha$$

$$\Rightarrow P\left(\frac{1}{b}\log\left(\frac{\alpha}{2}\right) < \log(1 - e^{-X}) < \frac{1}{b}\log\left(1 - \frac{\alpha}{2}\right)\right) = 1 - \alpha$$

$$\Rightarrow P\left(\left(\frac{\log(1 - e^{-X})}{\log\left(1 - \frac{\alpha}{2}\right)}\right)^{-1} < b < \left(\frac{\log(1 - e^{-X})}{\log\left(\frac{\alpha}{2}\right)}\right)^{-1}\right) = 1 - \alpha$$

Hence, a $100(1-\alpha)\%$ CI for b is

$$\left[\left(\frac{\log(1-e^{-X})}{\log(1-\frac{\alpha}{2})} \right)^{-1}, \left(\frac{\log(1-e^{-X})}{\log(\frac{\alpha}{2})} \right)^{-1} \right].$$