Example 11 Suppose $X_1, \ldots, X_n$ are IID with PDF $f(x) = \theta^2 e^{-\theta x}$, $x > 0$. Find the MLE of θ.

(i) $$L(\theta) = \prod_{i=1}^{n} \left[\theta^2 e^{-\theta x_i}\right]$$

$$= \theta^{2n} e^{-\theta \sum_{i=1}^{n} x_i}$$

(ii) $$\log L(\theta) = 2n \log \theta - \theta \sum_{i=1}^{n} x_i$$

(iii) $$\frac{d \log L(\theta)}{d\theta} = \frac{2n}{\theta} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\theta} = \frac{2n}{\sum_{i=1}^{n} x_i} = \frac{2}{\bar{x}}$$

(iv) $$\frac{d^2 \log L(\theta)}{d\theta^2} = -\frac{2n}{\theta^2} < 0$$

Hence, $\hat{\theta} = \frac{2}{\bar{x}}$ is the MLE of θ.

Example 12 Suppose $X_1, \ldots, X_n$ are IID with PDF $f(x) = \dfrac{\theta^k x^{k-1} e^{-\theta x}}{\Gamma(k)}$, $x > 0$. Assume k is known and θ is unknown. Find the MLE of θ.

(i) $$L(\theta) = \prod_{i=1}^{n} \left[\frac{\theta^k x_i^{k-1} e^{-\theta x_i}}{\Gamma(k)} \right]$$

$$= \frac{\theta^{nk}}{[\Gamma(k)]^n} \left(\prod_{i=1}^{n} x_i \right)^{k-1} e^{-\theta \sum_{i=1}^{n} x_i}$$

(ii) $$\log L(\theta) = nk \log \theta - n \log \Gamma(k) + (k-1) \sum_{i=1}^{n} \log x_i - \theta \sum_{i=1}^{n} x_i$$

(iii) $$\frac{d \log L(\theta)}{d\theta} = \frac{nk}{\theta} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\theta} = \frac{nk}{\sum_{i=1}^{n} x_i} = \frac{k}{\bar{x}}$$

(iv) $$\frac{d^2 \log L(\theta)}{d\theta^2} = -\frac{nk}{\theta^2} < 0.$$

Hence, $\hat{\theta} = \dfrac{k}{\bar{x}}$ is the MLE of θ.

<u>Example 13</u> Suppose $X_i \sim \text{Poisson}(i\lambda)$, $i = 1, 2, \ldots, n$ are independent. Find the MLE of λ.

(i) $$L(\lambda) = \prod_{i=1}^{n} \underbrace{\left[\frac{e^{-i\lambda}(i\lambda)^{x_i}}{x_i!} \right]}_{\text{PMF of } Po(i\lambda)}$$

$$= \frac{\boxed{\prod_{i=1}^{n} e^{-i\lambda}} \boxed{\prod_{i=1}^{n} i^{x_i}} \boxed{\prod_{i=1}^{n} \lambda^{x_i}}}{\prod_{i=1}^{n} x_i!}$$

$$= \frac{\boxed{e^{-\underline{(1+2+\cdots+n)}\lambda}} \boxed{\prod_{i=1}^{n} i^{x_i}} \boxed{\lambda^{x_1+\cdots+x_n}}}{\prod_{i=1}^{n} x_i!}$$

$$\boxed{1+2+\cdots+n = \frac{n(n+1)}{2}}$$

$$= \frac{e^{-\frac{n(n+1)}{2}\lambda} \prod_{i=1}^{n} i^{x_i} \lambda^{x_1+\cdots+x_n}}{\prod_{i=1}^{n} x_i!}$$

(ii) $$\log L(\lambda) = -\frac{n(n+1)\lambda}{2} + \sum_{i=1}^{n} x_i \log i$$

$$+ (x_1 + \cdots + x_n)\log\lambda - \sum_{i=1}^{n} \log x_i!$$

(iii) $$\frac{d \log L(\lambda)}{d\lambda} = -\frac{n(n+1)}{2} + 0 + \frac{x_1 + \cdots + x_n}{\lambda} + 0$$

$$= 0$$

$$\Rightarrow \quad \hat{\lambda} = \frac{2}{n(n+1)}\left(\sum_{i=1}^{n} x_i\right)$$

$$= \frac{2}{n+1} \times \bar{x}$$

(iv) $$\frac{d^2 \log L(\lambda)}{d\lambda^2} = -\frac{x_1 + \cdots + x_n}{\lambda^2} < 0$$

Hence, $\hat{\lambda} = \frac{2\bar{x}}{n+1}$ is the MLE.

Example 14 Suppose $X_1, \ldots, X_n$ are IID $LN(\theta, 1)$. Find the MLE of θ.

(i) $$L(\theta) = \prod_{i=1}^{n} \left[\frac{1}{x_i \sqrt{2\pi}} e^{-\frac{1}{2}(\log x_i - \theta)^2} \right]$$

$$= \frac{1}{\left(\prod_{i=1}^{n} x_i\right)(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\sum_{i=1}^{n}(\log x_i - \theta)^2}$$

(ii) $$\log L(\theta) = -\sum_{i=1}^{n} \log x_i - \frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(\log x_i - \theta)^2$$

(iii) $$\frac{d \log L(\theta)}{d\theta} = 0 + 0 - \frac{1}{\cancel{2}}\sum_{i=1}^{n} \cancel{2}(\log x_i - \theta)(-1)$$

$$= \sum_{i=1}^{n}(\log x_i - \theta)$$

$$= \left(\sum_{i=1}^{n} \log x_i\right) - \left(\sum_{i=1}^{n} \theta\right)$$

$$= \sum_{i=1}^{n} \log x_i - n\theta = 0$$

$$\Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} \log x_i$$

(iv) $$\frac{d^2 \log L(\theta)}{d\theta^2} = -n < 0$$

Hence, $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} \log x_i$ is the MLE.

Example 15 Suppose $X_1, \ldots, X_n$ are IID with CDF $F(x) = 1 - (1-x)^a$, $0 < x < 1$. Find the MLE of a.

(i)
$$L(a) = \prod_{i=1}^{n} \left. \frac{dF(x)}{dx} \right|_{x=x_i}$$
$$= \prod_{i=1}^{n} \left[a(1-x_i)^{a-1} \right]$$
$$= a^n \left[\prod_{i=1}^{n} (1-x_i) \right]^{a-1}.$$

(ii)
$$\log L(a) = n \log a + (a-1) \sum_{i=1}^{n} \log(1-x_i)$$

(iii)
$$\frac{d \log L(a)}{da} = \frac{n}{a} + \sum_{i=1}^{n} \log(1-x_i) = 0$$
$$\Rightarrow \hat{a} = -\frac{n}{\sum_{i=1}^{n} \log(1-x_i)}.$$

(iv)
$$\frac{d^2 \log L(a)}{da^2} = -\frac{n}{a^2} < 0$$

Hence, $\hat{a} = -\dfrac{n}{\sum_{i=1}^{n} \log(1-x_i)}$ is the MLE.

Example 16 Suppose $X_1, \ldots, X_n$ are IID $LN(\theta, \theta)$. Find the MLE of θ.

HOME WORK