Example 1 Suppose $X_1, \ldots, X_n$ are IID with PDF $\frac{1}{a} e^{-\frac{x}{a}}$, $x > 0$. Find the MLE of a.

(i) $$L(a) = \prod_{i=1}^{n} \left[\frac{1}{a} e^{-\frac{x_i}{a}} \right]$$

$$= \frac{1}{a^n} e^{-\frac{1}{a} \sum_{i=1}^{n} x_i}$$

(ii) $$\log L(a) = -n \log a - \frac{1}{a} \sum_{i=1}^{n} x_i$$

(iii) $$\frac{d \log L(a)}{da} = -\frac{n}{a} + \frac{1}{a^2} \sum_{i=1}^{n} x_i$$

$$= 0$$

$$\Rightarrow \hat{a} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$$

(iv) $$\frac{d^2 \log L(a)}{da^2} = \frac{n}{a^2} - \frac{2}{a^3} \sum_{i=1}^{n} x_i$$

$$= \frac{1}{a^3} \left[na - 2 \sum_{i=1}^{n} x_i \right]$$

$$\overset{a=\hat{a}}{=} \frac{1}{(\hat{a})^3} \left[n\hat{a} - 2 \sum_{i=1}^{n} x_i \right]$$

$$= \frac{1}{(\hat{a})^3} \left[\left(\sum_{i=1}^{n} x_i \right) - 2 \sum_{i=1}^{n} x_i \right]$$

$$< 0$$

$\Rightarrow$ Hence, $\hat{a} = \bar{x}$ is the MLE of a.

Example 2

Suppose $X_1, \ldots, X_n$ are IID with PDF $f(x) = a x^{a-1}$, $0 < x < 1$. Find the MLE of a.

(i) $$L(a) = \prod_{i=1}^{n} \left(a x_i^{a-1}\right) = a^n \left(\prod_{i=1}^{n} x_i^{a-1}\right) = a^n \left(\prod_{i=1}^{n} x_i\right)^{a-1}$$

(ii) $$\log L(a) = n \log a + (a-1) \log\left(\prod_{i=1}^{n} x_i\right) = n \log a + (a-1)\left(\sum_{i=1}^{n} \log x_i\right)$$

(iii) $$\frac{d \log L(a)}{da} = \frac{n}{a} + \sum_{i=1}^{n} \log x_i = 0$$

$$\Rightarrow \hat{a} = -\frac{n}{\sum_{i=1}^{n} \log x_i}$$

(iv) $$\frac{d^2 \log L(a)}{da^2} = -\frac{n}{a^2} < 0$$

$\Rightarrow \hat{a} = -\dfrac{n}{\sum_{i=1}^{n} \log x_i}$ is the MLE.

Example 3 Suppose $X_1, \ldots, X_n$ are IID Bin (m, p). Assume m is known but p is not. Find the MLE of p.

(i) $$L(p) = \prod_{i=1}^{n}\left[\binom{m}{x_i} p^{x_i} (1-p)^{m-x_i}\right]$$

$$= \left[\prod_{i=1}^{n}\binom{m}{x_i}\right]\left[\prod_{i=1}^{n} p^{x_i}\right]\left[\prod_{i=1}^{n}(1-p)^{m-x_i}\right]$$

$$= \left[\prod_{i=1}^{n}\binom{m}{x_i}\right] p^{\sum_{i=1}^{n} x_i} (1-p)^{\sum_{i=1}^{n}(m-x_i)}$$

$$= \left[\prod_{i=1}^{n}\binom{m}{x_i}\right] p^{\sum_{i=1}^{n} x_i} (1-p)^{mn-\left(\sum_{i=1}^{n} x_i\right)}$$

(ii) $$\log L(p) = \sum_{i=1}^{n} \log\binom{m}{x_i} + \left(\sum_{i=1}^{n} x_i\right)\log p$$

$$+ \left[mn - \left(\sum_{i=1}^{n} x_i\right)\right]\log(1-p)$$

(iii) $$\frac{d \log L(p)}{dp} = 0 + \frac{\sum_{i=1}^{n} x_i}{p} - \frac{mn - \left(\sum_{i=1}^{n} x_i\right)}{1-p}$$

$$= 0$$

$$\Rightarrow \hat{p} = \frac{1}{mn}\sum_{i=1}^{n} x_i$$

(iv) $$\frac{d^2 \log L(p)}{dp^2} = -\frac{\sum_{i=1}^{n} x_i}{p^2} - \frac{mn - \left(\sum_{i=1}^{n} x_i\right)}{(1-p)^2}$$

$$< 0$$

Hence, $\hat{p} = \frac{1}{mn} \sum_{i=1}^{n} x_i$ is the MLE.

Example 4 Suppose X is $Bin(m, p)$.
Assume m is known but p is not known.
Find the MLE of p.

(i) $$L(p) = \prod_{i=1}^{1} \binom{m}{x} p^{x} (1-p)^{m-x}$$

$$= \binom{m}{x} p^{x} (1-p)^{m-x}$$

(ii) $$\log L(p) = \log\binom{m}{x} + x\log p + (m-x)\log(1-p)$$

(iii) $$\frac{d\log L(p)}{dp} = \frac{x}{p} - \frac{m-x}{1-p} = 0$$

$$\Rightarrow \hat{p} = \frac{x}{m}$$

(iv) $$\frac{d^2\log L(p)}{dp^2} = -\frac{x}{p^2} - \frac{(m-x)}{(1-p)^2} < 0$$

Hence, $\boxed{\hat{p} = \frac{x}{m}}$ is the MLE of p

In ex 3, $\boxed{\hat{p} = \frac{1}{mn}\sum_{i=1}^{n} x_i}$

Example 5 Suppose $X_1, \dots, X_n$ are IID $LN(0, \theta^2)$. Find the MLE of θ.

(i) $$L(\theta) = \prod_{i=1}^{n} \left[\frac{1}{x_i \sqrt{2\pi}\, \theta} e^{-\frac{1}{2}\left(\frac{\log x_i}{\theta}\right)^2} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} \theta^n} \left(\prod_{i=1}^{n} \frac{1}{x_i} \right) e^{-\frac{1}{2\theta^2} \sum_{i=1}^{n} (\log x_i)^2}$$

(ii) $$\log L(\theta) = -\frac{n}{2} \log(2\pi) - n \log \theta - \sum_{i=1}^{n} \log x_1 - \frac{1}{2\theta^2} \sum_{i=1}^{n} (\log x_i)^2$$