Example 5

$$\hat{\theta} = \max(X_1, X_2).$$

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

$$= E(\underbrace{\max(X_1, X_2)}_{= Z}) - \theta$$

The CDF of Z is

$$F_Z(z) = P(Z \leq z)$$

$$= P(\max(X_1, X_2) \leq z)$$

$$= P(X_1 \leq z, X_2 \leq z)$$

$$\overset{\text{indep}}{=} P(X_1 \leq z)\, P(X_2 \leq z)$$

$$= \frac{z-\theta}{\theta+1-\theta} \times \frac{z-\theta}{\theta+1-\theta}$$

$$= (z-\theta)^2$$

The PDF of Z is $f_Z(z) = 2(z-\theta)$

$$= E(Z) - \theta$$

$$= \int_{\theta}^{\theta+1} z \cdot 2(z-\theta)\,dz - \theta$$

$$= 2\left[\frac{z^3}{3} - \frac{\theta z^2}{2}\right]_{\theta}^{\theta+1} - \theta$$

$$\neq 0$$

$\Rightarrow$ $\hat{\theta}$ is a biased estimator of θ.

Example 11 Suppose X is Bernoulli (p). Let X be an estimator of p. Find bias and MSE of X.

$$\text{Bias}(X) = E(X) - p$$

$$= p - p$$

$$= \boxed{0}$$

$\Rightarrow$ X is unbiased for p

$$\text{MSE}(X) = \text{Var}(X) + [\text{Bias}(X)]^2$$

$$= \text{Var}(X) + 0$$

$$= \boxed{p(1-p).}$$

Example 12 Suppose $X_1, \ldots, X_n$ are IID Bernoulli(p). Let $\bar{X}$ be an estimator of p. Find its bias and MSE.

$$\begin{aligned}
\text{Bias}(\bar{X}) &= E(\bar{X}) - p \\
&= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - p \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} E(X_i)\right) - p \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} p\right) - p \\
&= \frac{1}{n} \times np - p = 0
\end{aligned}$$

$\Rightarrow$ $\bar{X}$ is unbiased for p

$$\begin{aligned}
\text{MSE}(\bar{X}) &= \text{Var}(\bar{X}) + [\text{Bias}(\bar{X})]^2 \\
&= \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) + 0 \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} p(1-p) \\
&= \frac{1}{n^2} \times np(1-p) \\
&= \frac{p(1-p)}{n} \longrightarrow 0 \text{ as } n \to \infty
\end{aligned}$$

$\Rightarrow$ $\bar{X}$ is consistent for p

Example 13 Suppose X_1 and X_2 are IID $\text{Exp}\left(\frac{1}{\lambda}\right)$. Let $\boxed{\hat{\lambda} = \frac{4}{\pi}\sqrt{X_1 X_2}}$ be an estimator of λ. Find its bias and MSE.

$$\text{Bias}(\hat{\lambda}) = E(\hat{\lambda}) - \lambda$$

$$= E\left(\frac{4}{\pi}\sqrt{X_1 X_2}\right) - \lambda$$

$$= \frac{4}{\pi} E\left(\sqrt{X_1 X_2}\right) - \lambda$$

$$\overset{\text{indep}}{=} \frac{4}{\pi} \underline{E(\sqrt{X_1})}\ \underline{E(\sqrt{X_2})} - \lambda$$

If $X \sim \text{Exp}\left(\frac{1}{\lambda}\right)$ then

$$E(\sqrt{X}) = \int_0^\infty \sqrt{x}\, \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\, dx$$

$$= \frac{1}{\lambda} \int_0^\infty \sqrt{x}\, e^{-\frac{x}{\lambda}}\, dx$$

$$\boxed{\begin{aligned} \text{Set}\quad y &= \frac{x}{\lambda} \\ x &= \lambda y \\ dx &= \lambda\, dy \end{aligned}}$$

$$= \sqrt{\lambda} \int_0^\infty \sqrt{y}\, e^{-y}\, dy$$

$$= \sqrt{\lambda}\ \Gamma\left(\frac{3}{2}\right)$$

$$= \sqrt{\lambda}\ \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\lambda\pi}}{2}$$

$$= \frac{4}{\pi} \cdot \frac{\sqrt{\lambda\pi}}{2}\ \frac{\sqrt{\lambda\pi}}{2} - \lambda = 0$$

$\Rightarrow \hat{\lambda}$ is unbiased

$$\text{MSE}(\hat{\lambda}) = \text{Var}(\hat{\lambda}) + [\text{Bias}(\hat{\lambda})]^2$$

$$= \text{Var}\left(\frac{4}{\pi}\sqrt{X_1 X_2}\right) + 0$$

$$\boxed{\text{Var}(Z) = E(Z^2) - [E(Z)]^2}$$

$$= \frac{16}{\pi^2}\,\text{Var}\left(\sqrt{X_1 X_2}\right)$$

$$= \frac{16}{\pi^2}\left\{E(X_1 X_2) - \left[E\left(\sqrt{X_1 X_2}\right)\right]^2\right\}$$

$$\overset{\text{indep}}{=} \frac{16}{\pi^2}\left\{E(X_1)\,E(X_2) - \left[E\left(\sqrt{X_1}\right)E\left(\sqrt{X_2}\right)\right]^2\right\}$$

$$= \frac{16}{\pi^2}\left\{\lambda \cdot \lambda - \left[\frac{\sqrt{\lambda\pi}}{2} \cdot \frac{\sqrt{\lambda\pi}}{2}\right]^2\right\}$$

$$= \frac{16}{\pi^2}\left(\lambda^2 - \lambda^2\,\frac{\pi^2}{16}\right)$$

$$= \boxed{\lambda^2\left(\frac{16}{\pi^2} - 1\right)}.$$

Example 14

Suppose X and Y are independent RVs with

$$E(X) = 2\theta, \quad E(Y) = \theta$$
$$Var(X) = 4, \quad Var(Y) = 2.$$

Consider estimators

$$\widehat{\theta_1} = \frac{X}{4} + \frac{Y}{2}$$
$$\widehat{\theta_2} = X - Y.$$

Which of these 2 estimators is better?

$$\begin{aligned}
Bias(\widehat{\theta_1}) &= E(\widehat{\theta_1}) - \theta \\
&= E\left(\frac{X}{4} + \frac{Y}{2}\right) - \theta \\
&= \frac{1}{4}E(X) + \frac{1}{2}E(Y) - \theta \\
&= \frac{1}{4} \times 2\theta + \frac{1}{2} \times \theta - \theta \\
&= 0
\end{aligned}$$

$$\begin{aligned}
Bias(\widehat{\theta_2}) &= E(\widehat{\theta_2}) - \theta \\
&= E(X - Y) - \theta \\
&= E(X) - E(Y) - \theta \\
&= 2\theta - \theta - \theta \\
&= 0
\end{aligned}$$

In terms of bias both are equally good.

$$
\begin{aligned}
MSE(\hat{\theta}_1) &= Var(\hat{\theta}_1) \\
&= Var\left(\frac{X}{4} + \frac{Y}{2}\right) \\
&= \frac{1}{16} Var(X) + \frac{1}{4} Var(Y) \\
&= \frac{1}{16} \times 4 + \frac{1}{4} \times 2 \\
&= \boxed{\frac{3}{4}}
\end{aligned}
$$

$$
\begin{aligned}
MSE(\hat{\theta}_2) &= Var(\hat{\theta}_2) \\
&= Var(X - Y) \\
&= Var(X) + Var(Y) \\
&= 4 + 2 \\
&= \boxed{6}
\end{aligned}
$$

$\hat{\theta}_1$ is the better estimator.