Example 1 Suppose $X \sim Po(\lambda)$.
Let $\hat{\lambda} = X$ be an estimator.
Find bias and MSE of $\hat{\lambda}$.

$$\begin{aligned}
bias(\hat{\lambda}) &= E(\hat{\lambda}) - \lambda \\
&= E(X) - \lambda \\
&= \lambda - \lambda \\
&= 0
\end{aligned}$$

$\Rightarrow$ $\hat{\lambda}$ is unbiased for λ.

$$\begin{aligned}
MSE(\hat{\lambda}) &\overset{\text{prop } 1}{=} Var(\hat{\lambda}) + [Bias(\hat{\lambda})]^2 \\
&= Var(\hat{\lambda}) + 0 \\
&= Var(X) \\
&= \lambda.
\end{aligned}$$

Example 2 Suppose $X_1, \ldots, X_n$ are IID $Po(\lambda)$. Let $\hat{\lambda} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ be an estimator of λ. Find bias and MSE of $\hat{\lambda}$.

$$
\begin{aligned}
\text{Bias}(\hat{\lambda}) &= E(\hat{\lambda}) - \lambda \\
&= E(\bar{X}) - \lambda \\
&= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \lambda \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} E(X_i)\right) - \lambda \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} \lambda\right) - \lambda \\
&= \frac{1}{n} \times n\lambda - \lambda \\
&= 0
\end{aligned}
$$

$\Rightarrow \hat{\lambda}$ is unbiased for λ.

$$MSE(\hat{\lambda}) \overset{prop\ 1}{=} Var(\hat{\lambda}) + [Bias(\hat{\lambda})]^2$$

$$= Var(\hat{\lambda})$$

$$= Var(\bar{X})$$

$$= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \lambda$$

$$= \frac{1}{n^2} \times n\lambda$$

$$= \frac{\lambda}{n} \longrightarrow 0 \text{ as } n \to \infty$$

$\Rightarrow$ $\hat{\lambda}$ is consistent

$\Rightarrow$ $\hat{\lambda}$ is unbiased & consistent

Example 3 Suppose $X \sim Bin(m, p)$. Let $\hat{p} = X/m$ be an estimator of p. Find bias and MSE of $\hat{p}$.

$$\begin{aligned}
Bias(\hat{p}) &= E(\hat{p}) - p \\
&= E\left(\frac{X}{m}\right) - p \\
&= \frac{1}{m} E(X) - p \\
&= \frac{1}{\cancel{m}} \times \cancel{m}p - p \\
&= 0
\end{aligned}$$

$\Rightarrow \hat{p}$ is unbiased for p.

$$\begin{aligned}
MSE(\hat{p}) &= Var(\hat{p}) + [Bias(\hat{p})]^2 \\
&= Var(\hat{p}) \\
&= Var\left(\frac{X}{m}\right) \\
&= \frac{1}{m^2} Var(X) \\
&= \frac{1}{m^2} \times mp(1-p) \\
&= \frac{p(1-p)}{m}.
\end{aligned}$$

Example 4 Suppose $X_1, \ldots, X_n$ are IID $\mathrm{Exp}(a)$. Let $\hat{a} = \bar{X}$ be an estimator of a. Find bias and MSE of $\hat{a}$.

$$\begin{aligned}
\mathrm{Bias}(\hat{a}) &= E(\hat{a}) - a \\
&= E(\bar{X}) - a \\
&= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - a \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} E(X_i)\right) - a \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} \frac{1}{a}\right) - a \\
&= \frac{1}{n} \times \frac{n}{a} - a \\
&= \frac{1}{a} - a \neq 0
\end{aligned}$$

$\Rightarrow$ $\hat{a}$ is a biased estimator of a.

$$MSE(\hat{a}) \overset{\text{prop 1}}{=} Var(\hat{a}) + [Bias(\hat{a})]^2$$

$$= Var(\hat{a}) + \left(\frac{1}{a} - a\right)^2$$

$$= Var(\bar{X}) + \left(\frac{1}{a} - a\right)^2$$

$$= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) + \left(\frac{1}{a} - a\right)^2$$

$$= \left(\frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)\right) + \left(\frac{1}{a} - a\right)^2$$

$$= \left(\frac{1}{n^2}\sum_{i=1}^{n} \frac{1}{a^2}\right) + \left(\frac{1}{a} - a\right)^2$$

$$= \frac{1}{n^2} \times \frac{n}{a^2} + \left(\frac{1}{a} - a\right)^2$$

$$= \frac{1}{na^2} + \left(\frac{1}{a} - a\right)^2$$

$$\longrightarrow \left(\frac{1}{a} - a\right)^2 \quad \text{as} \quad n \to \infty$$

$$\neq 0$$

$\Rightarrow \hat{a}$ is not a consistent estimator

Example 5 Suppose $X_1, \ldots, X_n$ are IID with CDF $F(x)$ and PDF $f(x)$. Find CDF and PDF of $\max(X_1, \ldots, X_n)$.

Let $Z = \max(X_1, \ldots, X_n)$.

The CDF of Z is

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(\max(X_1, \ldots, X_n) \leq z) \\
&= P(X_1 \leq z, \ldots, X_n \leq z) \\
&\overset{\text{indep}}{=} P(X_1 \leq z) \cdots P(X_n \leq z) \\
&= F(z) \cdots F(z) \\
&= \boxed{[F(z)]^n}.
\end{aligned}$$

The PDF of Z is

$$\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_Z(z) \\
&= \frac{d}{dz} [F(z)]^n \\
&= \boxed{n [F(z)]^{n-1} f(z)}
\end{aligned}$$

Example 6 Suppose $X_1, \ldots, X_n$ are IID with CDF $F(x)$ and PDF $f(x)$. Find CDF and PDF of $\min(X_1, \ldots, X_n)$.

Let $Z = \min(X_1, \ldots, X_n)$.

The CDF of Z is

$$
\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= 1 - P(Z > z) \\
&= 1 - P(\min(X_1, \ldots, X_n) > z) \\
&= 1 - P(X_1 > z, \ldots, X_n > z) \\
&\overset{\text{indep}}{=} 1 - P(X_1 > z) \cdots P(X_n > z) \\
&= 1 - [1 - P(X_1 \leq z)] \cdots [1 - P(X_n \leq z)] \\
&= 1 - [1 - F(z)] \cdots [1 - F(z)] \\
&= \boxed{1 - [1 - F(z)]^n}.
\end{aligned}
$$

The PDF of Z is

$$
\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_Z(z) \\
&= \boxed{n\,[1 - F(z)]^{n-1} f(z)}
\end{aligned}
$$