Example I

Suppose $X_1, \dots, X_n$ are IID as $Po(\lambda)$. Find

$$P\left(a \leq \sum_{i=1}^{n} X_i \leq b\right)$$

$$= P\left(\frac{a}{n} \leq \frac{1}{n}\sum_{i=1}^{n} X_i \leq \frac{b}{n}\right)$$

$$= P\left(\frac{a}{n} \leq \overline{X} \leq \frac{b}{n}\right)$$

$$\boxed{\begin{aligned} \mu &= E(X_i) = \lambda \\ \sigma^2 &= Var(X_i) = \lambda \\ \sigma &= \sqrt{\lambda} \end{aligned}}$$

$$= P\left(\frac{\frac{a}{n}-\lambda}{\sqrt{\lambda/n}} \leq \frac{\overline{X}-\lambda}{\sqrt{\lambda}/\sqrt{n}} \leq \frac{\frac{b}{n}-\lambda}{\sqrt{\lambda/n}}\right)$$

$$\underset{CLT}{\overset{by}{\approx}} P\left(\frac{\frac{a}{n}-\lambda}{\sqrt{\lambda/n}} \leq N(0,1) \leq \frac{\frac{b}{n}-\lambda}{\sqrt{\lambda/n}}\right)$$

$$= P\left(N(0,1) \leq \frac{\frac{b}{n}-\lambda}{\sqrt{\lambda/n}}\right) - P\left(N(0,1) \leq \frac{\frac{a}{n}-\lambda}{\sqrt{\lambda/n}}\right)$$

$$= \Phi\left(\frac{\frac{b}{n}-\lambda}{\sqrt{\lambda/n}}\right) - \Phi\left(\frac{\frac{a}{n}-\lambda}{\sqrt{\lambda/n}}\right).$$

Example 2

Suppose $X_1, \dots, X_n$ are IID $N(0,1)$. Find a and b such that

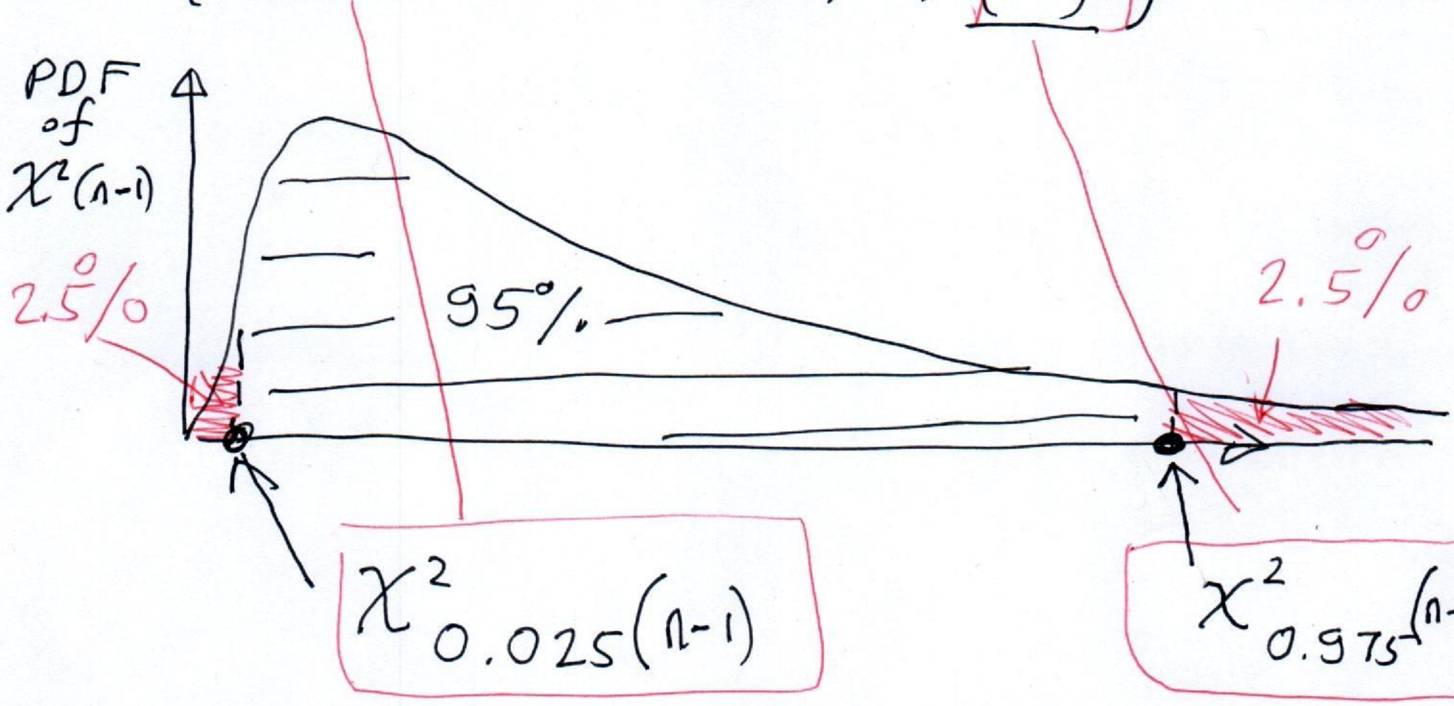
$$P\left(a \le S^2 \le b\right) = 0.95$$

where $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$.

$$\Rightarrow P\left(a \le \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \le b\right) = 0.95$$

$$\Rightarrow P\left((n-1)a \le \sum_{i=1}^{n} (X_i - \bar{X})^2 \le (n-1)b\right) = 0.95$$

$$\overset{\text{prop 3}}{\Rightarrow} P\left(\boxed{(n-1)a} \le \chi^2(n-1) \le \boxed{(n-1)b}\right) = 0.95$$



$$(n-1)a = \chi^2_{0.025}(n-1) \Rightarrow a = \frac{1}{n-1}\chi^2_{0.025}(n-1)$$

$$(n-1)b = \chi^2_{0.975}(n-1) \Rightarrow b = \frac{1}{n-1}\chi^2_{0.975}(n-1)$$

Example 3

Suppose $X_1, \dots, X_n$ are IID $N(0,1)$.
Find $P(\bar{X} < S)$

$$P(\bar{X} < S)$$

$$= P\left(\frac{\bar{X}}{S} < 1\right)$$

$$= P\left(\frac{\bar{X}}{S/\sqrt{n}} < \frac{1}{1/\sqrt{n}}\right)$$

$$\overset{\text{prop 4}}{=} P\left(t(n-1) < \sqrt{n}\right)$$

$$= \boxed{F_{t(n-1)}}(\sqrt{n})$$

denotes CDF of $t(n-1)$

Example 4 Let $X \sim \chi^2(a)$.
Show that $E(X) = a$.

$$E(X) = \int_0^\infty x \cdot \frac{1}{2^{\frac{a}{2}} \Gamma\left(\frac{a}{2}\right)} x^{\frac{a}{2}-1} e^{-\frac{x}{2}} dx$$

$$= \frac{1}{2^{\frac{a}{2}} \Gamma\left(\frac{a}{2}\right)} \int_0^\infty x^{\frac{a}{2}} e^{-\frac{x}{2}} dx$$

$$\boxed{\begin{aligned} &\text{Set } y = \frac{x}{2} \\ &\Rightarrow x = 2y \\ &\Rightarrow dx = 2dy \end{aligned}}$$

$$= \frac{1}{2^{\frac{a}{2}} \Gamma\left(\frac{a}{2}\right)} \int_0^\infty (2y)^{\frac{a}{2}} e^{-y} \, 2dy$$

$$= \frac{2^{\frac{a}{2}+1}}{2^{\frac{a}{2}} \Gamma\left(\frac{a}{2}\right)} \int_0^\infty y^{\frac{a}{2}} e^{-y} dy$$

$$= \frac{2}{\Gamma\left(\frac{a}{2}\right)} \Gamma\left(\frac{a}{2}+1\right) = \frac{2}{\cancel{\Gamma\left(\frac{a}{2}\right)}} \frac{a}{2} \cancel{\Gamma\left(\frac{a}{2}\right)}$$

$$= a.$$

Example 5

Let $X \sim t(a)$. Show that $E(X) = 0$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\,\Gamma\left(\frac{a}{2}\right)} \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}} dx$$

$$= \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\,\Gamma\left(\frac{a}{2}\right)} \int_{-\infty}^{\infty} x \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}} dx$$

$$= \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\,\Gamma\left(\frac{a}{2}\right)} \left[\int_{0}^{\infty} x \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}} dx + \underbrace{\int_{-\infty}^{0} x \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}} dx}_{\text{set } \boxed{\begin{array}{c} y = -x \\ x = -y \\ dx = -dy \end{array}}}\right]$$

$$= \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\,\Gamma\left(\frac{a}{2}\right)} \left[\int_{0}^{\infty} \cancel{x \left(1 + \frac{x^2}{a}\right)^{-\frac{a+1}{2}}} dx - \int_{0}^{\infty} \cancel{y \left(1 + \frac{y^2}{a}\right)^{-\frac{a+1}{2}}} dy\right]$$

$$= 0$$