A2, Exam 2018 / 2019

(b) $\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$

$= E\left(n \min(X_1, \dots, X_n)\right) - \theta$

$= n\, E\left(\min(X_1, \dots, X_n)\right) - \theta$

Let $Z = \min(X_1, \dots, X_n)$

The CDF of Z is

$$F_Z(z) = P(Z \leq z)$$

$$= 1 - P(Z > z)$$

$$= 1 - P\left(\underline{\min(X_1, \dots, X_n)} > z\right)$$

$$= 1 - P(X_1 > z, \dots, X_n > z)$$

$$\underset{=}{\text{indep}}\; 1 - P(X_1 > z) \dots P(X_n > z)$$

$$= 1 - \left(P(X_1 > z)\right)^n$$

$$= 1 - \left(1 - \underline{P(X_1 \leq z)}\right)^n$$

$$= 1 - \left(\cancel{1} - \left(\cancel{1} - e^{-\frac{z}{\theta}}\right)\right)^n$$

$$= 1 - e^{-\frac{nz}{\theta}}$$

The PDF of Z is

$$f_Z(z) = \boxed{\frac{n}{\theta} e^{-\frac{nz}{\theta}}} \leftarrow \text{PDF of } Exp\left(\frac{n}{\theta}\right)$$

$$\text{Bias}(\hat{\theta}) = nE(Z) - \theta$$

$$= n \times \frac{\theta}{n} - \theta$$

$$= 0$$

$\Rightarrow \hat{\theta}$ is unbiased.

(ii) $\text{MSE}(\hat{\theta})$

$= \text{Var}(\hat{\theta})$ because bias $= 0$

$= \text{Var}(nZ)$

$= n^2 \, \text{Var}(Z)$

$= \cancel{n^2} \cdot \frac{\theta^2}{\cancel{n^2}}$

$= \theta^2$

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(i)

$$L(\sigma^2) = \prod_{i=1}^{n} \left[\frac{x_i}{\sigma^2} \exp\left(-\frac{x_i^2}{2\sigma^2}\right) \right]$$

$$= \frac{1}{\sigma^{2n}} \left(\prod_{i=1}^{n} x_i \right) \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2 \right)$$

(ii)

$$\log L(\sigma^2) = -2n \log \sigma + \sum_{i=1}^{n} \log x_i$$

$$- \frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2$$

$$\frac{d \log L(\sigma^2)}{d\sigma} = -\frac{2n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} x_i^2 = 0$$

$$\Rightarrow \widehat{\sigma^2} = \frac{1}{2n} \sum_{i=1}^{n} x_i^2$$

$$\frac{d^2 \log L(\sigma^2)}{d\sigma^2} = + \frac{2n}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=1}^{n} x_i^2$$

$$= \frac{1}{\sigma^4} \left[2n\sigma^2 - 3 \sum_{i=1}^{n} x_i^2 \right]$$

$$\overset{\sigma^2 = \widehat{\sigma^2}}{=} \frac{1}{\widehat{\sigma^4}} \left[\sum_{i=1}^{n} x_i^2 - 3 \sum_{i=1}^{n} x_i^2 \right]$$

$$< 0$$

Hence $\widehat{\sigma^2}$ is an MLE.

(iii) $\sigma \longrightarrow \sigma^2$ is a 1-to-1 function

By the invariance principle,

$$\hat{\sigma} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2}.$$

(iv)

$$\widehat{\sigma^2} = \frac{1}{2n}\sum_{i=1}^{n} X_i^2$$

$$\text{Bias}(\widehat{\sigma^2}) = E(\widehat{\sigma^2}) - \sigma^2$$

$$= E\left(\frac{1}{2n}\sum_{i=1}^{n} X_i^2\right) - \sigma^2$$

$$= \frac{1}{2n}\sum_{i=1}^{n} \underline{\underline{E(X_i^2)}} - \sigma^2$$

$$= \frac{1}{2n}\sum_{i=1}^{n}\int_0^\infty x^2 \cdot \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)dx - \sigma^2$$

$$= \frac{1}{2n\sigma^2}\sum_{i=1}^{n}\int_0^\infty x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right)dx - \sigma^2$$

$$\boxed{\begin{aligned} \text{Set} \quad y &= \frac{x^2}{2\sigma^2} \\ x &= \sqrt{2}\,\sigma\sqrt{y} \\ \frac{dx}{dy} &= \frac{\sigma}{\sqrt{2y}} \end{aligned}}$$

$$= \frac{1}{2n\sigma^2}\sum_{i=1}^{n}\int_0^\infty (\sqrt{2}\,\sigma)^3\, y^{\frac{3}{2}} e^{-y}\,\frac{\sigma}{\sqrt{2}\sqrt{y}}\,dy - \sigma^2$$

$$= \frac{1}{\cancel{2} n \sigma^2} \cancel{2} \sigma^4 \sum_{i=1}^{n} \boxed{\int_0^\infty y e^{-y} \, dy} - \sigma^2$$

$$= \frac{\sigma^2}{n} \sum_{i=1}^{n} \underbrace{\boxed{\Gamma(2)}}_{= 1} - \sigma^2$$

$$= \frac{\sigma^2}{n} \times n - \sigma^2$$

$$= 0$$

$\Rightarrow$ Hence, $\widehat{\sigma^2}$ is unbiased.

B5, 2018/19 Exam

$$\hat{\theta}_1 = a(X_1 + X_2)$$

$$\hat{\theta}_2 = b\sqrt{X_1 X_2}$$

(i) $\text{Bias}(\hat{\theta}_1) = E(\hat{\theta}_1) - \lambda$

$$= E(a(X_1 + X_2)) - \lambda$$

$$= a[E(X_1) + E(X_2)] - \lambda$$

$$= a[\lambda + \lambda] - \lambda$$

$$= 2a\lambda - \lambda$$

$$= (2a - 1)\lambda$$

$$= 0 \quad \text{if} \quad a = \tfrac{1}{2}.$$

(ii) $\text{Bias}(\hat{\theta}_2) = E(\hat{\theta}_2) - \lambda$

$$= E(b\sqrt{X_1 X_2}) - \lambda$$

$$\overset{\text{indep}}{=} b\, E(\sqrt{X_1})\, E(\sqrt{X_2}) - \lambda$$

$$= b[E(\sqrt{X_1})]^2 - \lambda$$

$$= b\left[\int_0^\infty \underline{\sqrt{x}\,\tfrac{1}{\lambda} e^{-\frac{x}{\lambda}}\, dx}\right]^2 - \lambda$$

$$\boxed{\text{set } y = \tfrac{x}{\lambda} \Rightarrow x = \lambda y \Rightarrow dx = \lambda\, dy}$$

$$= b\left[\int_0^\infty \sqrt{\lambda}\sqrt{y}\,\tfrac{1}{\cancel{\lambda}} e^{-y}\, \cancel{\lambda}\, dy\right]^2 - \lambda$$

$$= b\left[\sqrt{\lambda}\,\boxed{\int_0^\infty \sqrt{y}\,e^{-y}\,dy}\right]^2 - \lambda$$

$$= b\left[\sqrt{\lambda}\,\Gamma\left(\tfrac{3}{2}\right)\right]^2 - \lambda$$

$$= b\left[\sqrt{\lambda}\,\tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)\right]^2 - \lambda$$

$$= b\left[\sqrt{\lambda}\,\tfrac{1}{2}\,\sqrt{\pi}\right]^2 - \lambda$$

$$= b\,\frac{\lambda\,\pi}{4} - \lambda$$

$$= 0 \quad \text{if} \quad b = \frac{4}{\pi}.$$

(iii) $$\mathrm{Var}(\hat{\theta}_1) = \mathrm{Var}(a(X_1 + X_2))$$

$$= a^2\left[\mathrm{Var}(X_1) + \mathrm{Var}(X_2)\right]$$

$$= a^2\left[\lambda^2 + \lambda^2\right]$$

$$= 2a^2\lambda^2$$

(iv) $$\mathrm{Var}(\hat{\theta}_2) = E(\hat{\theta}_2^2) - \left[E(\hat{\theta}_2)\right]^2$$

$$= E(b^2 X_1 X_2) - \lambda^2$$

$$= \left(\frac{4}{\pi}\right)^2 E(X_1)\,E(X_2) - \lambda^2$$

$$= \left(\frac{4}{\pi}\right)^2 \lambda\,\lambda - \lambda^2$$

$$= \left[\left(\frac{4}{\pi}\right)^2 - 1\right]\lambda^2$$

$$F_Z(z) = 1 - e^{n\theta - nz}$$

$$= \frac{\alpha}{2}$$

$$\Rightarrow z = \theta - \frac{1}{n}\log\left(1 - \frac{\alpha}{2}\right)$$

$$F_Z(z) = 1 - \frac{\alpha}{2}$$

$$\Rightarrow z = \theta - \frac{1}{n}\log\left(\frac{\alpha}{2}\right)$$

$$P\left(\theta - \frac{1}{n}\log\left(1-\frac{\alpha}{2}\right) < Z < \theta - \frac{1}{n}\log\left(\frac{\alpha}{2}\right)\right)$$

$$= 1 - \alpha$$

Rearrage in terms of θ.