Example 10 Suppose $X_1, \ldots, X_n$ are IID $N(\mu, 1)$. Consider testing

$$H_0 : \mu = 0$$
$$\text{vs} \quad H_1 : \mu = 1.$$

Reject H_0 if $\boxed{\bar{X} > k}$. Find α and β.

$$\begin{aligned}
\alpha &= P(\text{Type I error}) \\
&= P(\text{Reject } H_0 \mid \mu = 0) \\
&= P(\bar{X} > k \mid \mu = 0) \\
&= P\left(\boxed{\frac{\bar{X} - 0}{1/\sqrt{n}}} > \frac{k - 0}{1/\sqrt{n}} \;\Big|\; \mu = 0\right) \\
&\overset{\text{prop } 6}{=} P(N(0,1) > \sqrt{n}\, k) \\
&= 1 - P(N(0,1) \leq \sqrt{n}\, k) \\
&= 1 - \Phi(\sqrt{n}\, k).
\end{aligned}$$

$$\begin{aligned}
\beta &= P(\text{Type II error}) \\
&= P(\text{Accept } H_0 \mid \mu = 1) \\
&= P(\bar{X} \leq k \mid \mu = 1) \\
&= P\left(\boxed{\frac{\bar{X} - 1}{1/\sqrt{n}}} \leq \frac{k - 1}{1/\sqrt{n}} \;\Big|\; \mu = 1\right) \\
&= P(N(0,1) \leq \sqrt{n}(k-1)) \\
&= \Phi(\sqrt{n}\,(k-1)).
\end{aligned}$$

Example 11 Suppose $X_1, \ldots, X_n$ are IID $N(\mu, 1)$. Consider testing

$$H_0 : \mu = 0$$
$$\text{vs} \quad H_1 : \mu \neq 0$$

Reject H_0 if $\boxed{|\bar{X}| > k}$. Find α and β.

$$\begin{aligned}
\alpha &= P(\text{Reject } H_0 \mid \mu = 0) \\
&= P(|\bar{X}| > k \mid \mu = 0) \\
&= P(\bar{X} > k \text{ OR } \bar{X} < -k \mid \mu = 0) \\
&= P(\bar{X} > k \mid \mu = 0) + P(\bar{X} < -k \mid \mu = 0) \\
&= P\left(\boxed{\frac{\bar{X} - 0}{1/\sqrt{n}}} > \frac{k - 0}{1/\sqrt{n}} \,\Big|\, \mu = 0\right) \\
&\quad + P\left(\boxed{\frac{\bar{X} - 0}{1/\sqrt{n}}} < \frac{-k - 0}{1/\sqrt{n}} \,\Big|\, \mu = 0\right) \\
&\overset{\text{prop 6}}{=} P(N(0,1) > \sqrt{n}\,k) + P(N(0,1) < -\sqrt{n}\,k) \\
&= 1 - P(N(0,1) \leq \sqrt{n}\,k) + P(N(0,1) < -\sqrt{n}\,k) \\
&= 1 - \Phi(\sqrt{n}\,k) + \Phi(-\sqrt{n}\,k)
\end{aligned}$$

$$\beta = P(\text{Accept } H_0 \mid \mu \neq 0)$$
$$= P(|\bar{X}| \leqslant k \mid \mu \neq 0)$$
$$= P(-k \leqslant \bar{X} \leqslant k \mid \mu \neq 0)$$
$$= P\left(\frac{-k-\mu}{1/\sqrt{n}} \leqslant \boxed{\frac{\bar{X}-\mu}{1/\sqrt{n}}} \leqslant \frac{k-\mu}{1/\sqrt{n}} \,\middle|\, \mu \neq 0\right)$$
$$\overset{\text{prop 6}}{=} P\left(-\sqrt{n}(k+\mu) \leqslant N(0,1) \leqslant \sqrt{n}(k-\mu)\right)$$
$$= \Phi\left(\sqrt{n}(k-\mu)\right) - \Phi\left(-\sqrt{n}(k+\mu)\right).$$

Example 12 Suppose $X_1, \dots, X_n$ are IID $N(0, \sigma^2)$. Consider testing

$$H_0 : \sigma^2 = 1$$
$$\text{vs } H_1 : \sigma^2 \neq 1.$$

Reject H_0 if $\boxed{S^2} > k$. Find α and β

sample variance $\boxed{S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}$

$$\alpha = P(\text{Reject } H_0 \mid \sigma^2 = 1)$$
$$= P(S^2 > k \mid \sigma^2 = 1)$$
$$= P\left(\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 > k \mid \sigma^2 = 1\right)$$
$$= P\left(\sum_{i=1}^{n} (X_i - \bar{X})^2 > k(n-1) \,\Big|\, \sigma^2 = 1\right)$$
$$\overset{\text{prop 3}}{=} P(\chi^2(n-1) > k(n-1))$$
$$= 1 - P(\chi^2(n-1) \leq k(n-1))$$
$$= 1 - \boxed{F_{\chi^2(n-1)}}(k(n-1))$$

CDF of $\chi^2(n-1)$.

$$\beta = P(\text{Accept } H_0 \mid \sigma^2 \neq 1)$$

$$= P(S^2 \leq k \mid \sigma^2 \neq 1)$$

$$= P\left(\frac{1}{(n-1)}\sum_{i=1}^{n}(X_i - \bar{X})^2 \leq k \;\middle|\; \sigma^2 \neq 1\right)$$

$$= P\left(\sum_{i=1}^{n}(X_i - \bar{X})^2 \leq (n-1)k \;\middle|\; \sigma^2 \neq 1\right)$$

$$\overset{\text{prop 3}}{=} P\left(\boxed{\sigma^2}\,\chi^2(n-1) \leq (n-1)k\right)$$

$$= P\left(\chi^2(n-1) \leq \frac{(n-1)k}{\sigma^2}\right)$$

$$= F_{\chi^2(n-1)}\left(\frac{(n-1)\,k}{\sigma^2}\right).$$

Example 13 Suppose $X_1, \dots, X_n$ are IID $N(\mu, \sigma^2)$ where σ^2 is unknown.

Consider testing

$$H_0 : \mu = 0$$
$$\text{vs } H_1 : \mu \neq 0.$$

Reject H_0 if $\boxed{\frac{\bar{X}}{S} > k}$. Find α and β.

$$\alpha = P(\text{Reject } H_0 \mid \mu = 0)$$
$$= P\left(\frac{\bar{X}}{S} > k \;\middle|\; \mu = 0\right)$$
$$= P\left(\frac{\bar{X}}{S/\sqrt{n}} > \frac{k}{1/\sqrt{n}} \;\middle|\; \mu = 0\right)$$
$$\overset{\text{prop 4}}{=} P\left(t(n-1) > \sqrt{n}\,k\right)$$
$$= 1 - P\left(t(n-1) \leq \sqrt{n}\,k\right)$$
$$= 1 - F_{t(n-1)}\left(\sqrt{n}\,k\right)$$

$$\beta = P(\text{Accept } H_0 \mid \mu \neq 0)$$
$$= P\left(\frac{\bar{X}}{S} \leq k \;\middle|\; \mu \neq 0\right)$$
$$= P\left(\bar{X} \leq Sk \mid \mu \neq 0\right)$$
$$= P\left(\bar{X} - \mu \leq Sk - \mu \mid \mu \neq 0\right)$$
$$= P\left(\boxed{\frac{\bar{X} - \mu}{S/\sqrt{n}}} \leq \frac{Sk - \mu}{S/\sqrt{n}} \;\middle|\; \mu \neq 0\right)$$

$$\overset{\text{prop 4}}{=} P\left(t(n-1) \le \frac{Sk-\mu}{S/\sqrt{n}} \right)$$

$$= F_{t(n-1)}\left(\frac{Sk-\mu}{S/\sqrt{n}} \right).$$

Home work

Suppose $X_1, \ldots, X_n$ are IID $N(\mu, \sigma^2)$ where σ^2 is unknown.
Consider testing

$$H_0 : \mu = 0$$
$$\text{vs} \quad H_1 : \mu \neq 0$$

Reject H_0 if $\frac{|\bar{X}|}{S} > k$. Find α.