Example I

Suppose $X \sim \text{Bernoulli}(p)$.

Consider testing

$$H_0 : p = \frac{1}{2}$$

$$\text{vs } H_1 : p \neq \frac{1}{2}$$

Reject H_0 if $X = 1$. Find α and β.

$$\begin{aligned}
\alpha &= P(\text{Type I error}) \\
&= P(\text{Reject } H_0 \mid H_0 \text{ true}) \\
&= P\left(X = 1 \mid p = \tfrac{1}{2}\right) \\
&= \boxed{\tfrac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\beta &= P(\text{Type II error}) \\
&= P(\text{Accept } H_0 \mid H_0 \text{ false}) \\
&= P\left(\text{Accept } H_0 \mid p \neq \tfrac{1}{2}\right) \\
&= 1 - P\left(\text{Reject } H_0 \mid p \neq \tfrac{1}{2}\right) \\
&= 1 - P\left(X = 1 \mid p \neq \tfrac{1}{2}\right) \\
&= \boxed{1 - p}, \quad p \neq \tfrac{1}{2}
\end{aligned}$$

Example 2 Suppose $X_1, \ldots, X_n$ are IID $Po(\lambda)$. Consider testing

$$H_0 : \lambda = 1$$
$$H_1 : \lambda \neq 1$$

Reject H_0 if $\overline{X} > 1$. Find α and β.

$$\begin{aligned}
\alpha &= P(\text{Type I error}) \\
&= P(\text{Reject } H_0 \mid H_0 \text{ true}) \\
&= P(\overline{X} > 1 \mid \lambda = 1) \\
&= P\left(\frac{1}{n}\sum_{i=1}^{n} X_i > 1 \,\middle|\, \lambda = 1\right) \\
&= P\left(\sum_{i=1}^{n} X_i > n \,\middle|\, \lambda = 1\right) \\
&= P(Po(\lambda n) > n \mid \lambda = 1) \\
&= P(Po(n) > n) \\
&= P(Po(n) \geq n+1) \\
&= \sum_{i=n+1}^{\infty} P(Po(n) = i) \\
&= \boxed{\sum_{i=n+1}^{\infty} \frac{e^{-n} n^{i}}{i!}}
\end{aligned}$$

$$
\begin{aligned}
\beta &= P(\text{Type II error}) \\
&= P(\text{Accept } H_0 \mid H_1 \text{ true}) \\
&= P(\bar{X} \leqslant 1 \mid \lambda \neq 1) \\
&= P\left(\frac{1}{n}\sum_{i=1}^{n} X_i \leqslant 1 \mid \lambda \neq 1\right) \\
&= P\left(\sum_{i=1}^{n} X_i \leqslant n \mid \lambda \neq 1\right) \\
&= P(Po(\lambda n) \leqslant n \mid \lambda \neq 1) \\
&= P(Po(\lambda n) \leqslant n), \quad \lambda \neq 1 \\
&= \sum_{i=0}^{n} P(Po(\lambda n) = i), \quad \lambda \neq 1 \\
&= \boxed{\sum_{i=0}^{n} \frac{e^{-\lambda n} (\lambda n)^i}{i!}, \quad \lambda \neq 1}
\end{aligned}
$$

Example 3 Suppose $X_1, \ldots, X_n$ are IID $N(\mu, 1)$. Consider testing $H_0 : \mu = 0$ vs $H_1 : \mu \neq 0$. Reject H_0 if $|\bar{X}| > 1$. Find α and β.

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ true})$$

$$= P(|\bar{X}| > 1 \mid \mu = 0)$$

$$= P(\bar{X} > 1 \ \underline{\text{OR}}\ \bar{X} < -1 \mid \mu = 0)$$

$$= P(\bar{X} > 1 \mid \mu = 0) + P(\bar{X} < -1 \mid \mu = 0)$$

$$= P\left(\boxed{\frac{\bar{X} - 0}{1/\sqrt{n}}} > \frac{1-0}{1/\sqrt{n}} \mid \mu = 0\right) + P\left(\boxed{\frac{\bar{X} - 0}{1/\sqrt{n}}} < \frac{-1-0}{1/\sqrt{n}} \mid \mu = 0\right)$$

$$\overset{\text{prop } 6}{=} P(N(0,1) > \sqrt{n}) + P(N(0,1) < -\sqrt{n})$$

$$= 1 - P(N(0,1) \leq \sqrt{n}) + P(N(0,1) < -\sqrt{n})$$

$$= \boxed{1 - \Phi(\sqrt{n}) + \Phi(-\sqrt{n}).}$$

$$\beta = P(\text{Accept } H_0 \mid H_1 \text{ true})$$

$$= P(|\bar{X}| \leq 1 / \mu \neq 0)$$

$$= P(-1 \leq \bar{X} \leq 1 \mid \mu \neq 0)$$

$$= P\left(\frac{-1-\mu}{1/\sqrt{n}} \leq \frac{\bar{X}-\mu}{1/\sqrt{n}} \leq \frac{1-\mu}{1/\sqrt{n}} \,\middle|\, \mu \neq 0 \right)$$

$$\underset{\text{prop } 6}{=} P\left(\frac{-1-\mu}{1/\sqrt{n}} \leq N(0,1) \leq \frac{1-\mu}{1/\sqrt{n}} \right), \quad \mu \neq 0$$

$$= \boxed{\Phi\left(\frac{1-\mu}{1/\sqrt{n}}\right) - \Phi\left(\frac{-1-\mu}{1/\sqrt{n}}\right), \quad \mu \neq 0}$$