Example 1

Suppose $X_1, \ldots, X_n$ is a random sample from

$$p(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}$$

Find $E(\bar{x})$ and $Var(\bar{x})$.

$$\mu = E(X) = a \times 1 + 0$$
$$= a$$

$$\Rightarrow \quad E(\bar{X}) = a$$

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$$
$$= (a^2 \times 1 + 0) - a^2$$
$$= 0$$

$$\Rightarrow \quad Var(\bar{X}) = \frac{0}{n} = 0.$$

Example 2 Suppose $X_1, \ldots, X_n$ is a random sample from a distribution with PDF $f(x) = a x^{a-1}, \; 0 < x < 1,$ $a > 0$. Find $E(\bar{X})$ and $Var(\bar{X})$.

$$\mu = E(X) = \int_0^1 x \cdot a x^{a-1} \, dx$$

$$= a \int_0^1 x^a \, dx$$

$$= a \left[\frac{x^{a+1}}{a+1} \right]_0^1$$

$$= a \left[\frac{1}{a+1} - 0 \right]$$

$$= \frac{a}{a+1}.$$

$$\Rightarrow \quad E(\bar{X}) = \frac{a}{a+1}.$$

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$$

$$= \int_0^1 x^2 \cdot a x^{a-1} dx - \left(\frac{a}{a+1}\right)^2$$

$$= a \int_0^1 x^{a+1} dx - \left(\frac{a}{a+1}\right)^2$$

$$= a \left[\frac{x^{a+2}}{a+2}\right]_0^1 - \left(\frac{a}{a+1}\right)^2$$

$$= a\left(\frac{1}{a+2} - 0\right) - \left(\frac{a}{a+1}\right)^2$$

$$= \frac{a}{a+2} - \left(\frac{a}{a+1}\right)^2$$

$$\Rightarrow Var(\bar{X}) = \frac{1}{n}\left[\frac{a}{a+2} - \left(\frac{a}{a+1}\right)^2\right].$$

Example 3 Suppose $X_1, \dots, X_n$ is a random sample from Bernoulli (p).

Find $P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$

$X \sim$ Bernoulli(p) if

$$X = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{" " } 1-p \end{cases}$$

The PMF of X is

$$p(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$$

$$= p^x (1-p)^{1-x}$$

$$\overset{\text{indep}}{=} P(X_1 = x_1) \cdots P(X_n = x_n)$$

$$= p^{x_1} (1-p)^{1-x_1} \cdots p^{x_n} (1-p)^{1-x_n}$$

$$= p^{x_1 + \cdots + x_n} (1-p)^{1-x_1 + \cdots + 1-x_n}$$

$$= p^{\sum_{i=1}^{n} x_i} (1-p)^{n - \sum_{i=1}^{n} x_i}$$

Example 4 Suppose $X_1, \dots, X_n$ is a random sample from $Exp(\lambda)$. Find $P(X_1 = x_1, \dots, X_n = x_n)$.

$$\stackrel{\text{indep}}{=} P(X_1 = x_1) \cdots P(X_n = x_n)$$

$$= \lambda e^{-\lambda x_1} \cdots \lambda e^{-\lambda x_n}$$

$$= \lambda^n e^{-\lambda x_1 - \cdots - \lambda x_n}$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$