SOLUTIONS TO
MATH10282
INTRO TO STATISTICS
RESIT EXAM

Solutions to Question 1

ILOs addressed: present numerical summaries of a data set.

Suppose that we have the following sample of observations

1.75, 0.77, 0.39, 0.34, 0.07, 0.58, -0.54, -2.78, 0.75, -0.68

The sample mean is

$$\frac{1}{10}(1.75 + 0.77 + 0.39 + 0.34 + 0.07 + 0.58 - 0.54 - 2.78 + 0.75 - 0.68) = 0.065$$

(1 marks)

UNSEEN

The sample variance is

$$\frac{1}{9}\left[(1.75 - \overline{x})^2 + (0.77 - \overline{x})^2 + \cdots + (-0.68 - \overline{x})^2\right] = 1.474117$$

(1 marks)

UNSEEN

Arrange the data as

-2.78, -0.68, -0.54, 0.07, 0.34, 0.39, 0.58, 0.75, 0.77, 1.75

The middle two numbers are 0.34 and 0.39. The median is their average which is 0.365. (1 marks)

UNSEEN

Note that $r = 2.75$ and $r' = 3$, so $Q(1/4) = x_{(2)} + 0.75\left(x_{(3)} - x_{(2)}\right) = -0.575$. (1 marks)

UNSEEN

Note that $r = 8.25$ and $r' = 8$, so $Q(3/4) = x_{(8)} + 0.25\left(x_{(9)} - x_{(8)}\right) = 0.755$. (1 marks)

UNSEEN

The range of the data are

1.75 - (-2.78) = 4.53.

(1 marks)

UNSEEN

Note that $r = p(n+1)$ and $r^{'} = [p(n+1)]$ are

$$r = \begin{cases} 3m + \frac{3}{4}, & \text{if } n = 4m, \\ 3m, & \text{if } n = 4m-1, \\ 3m - \frac{3}{4}, & \text{if } n = 4m-2, \\ 3m - \frac{6}{4}, & \text{if } n = 4m-3 \end{cases}$$

and

$$r^{'} = \begin{cases} 3m, & \text{if } n = 4m, \\ 3m, & \text{if } n = 4m-1, \\ 3m-1, & \text{if } n = 4m-2, \\ 3m-2, & \text{if } n = 4m-3, \end{cases}$$

respectively. So,

$$r - r^{'} = \begin{cases} \frac{3}{4}, & \text{if } n = 4m, \\ 0, & \text{if } n = 4m-1, \\ \frac{1}{4}, & \text{if } n = 4m-2, \\ \frac{1}{2}, & \text{if } n = 4m-3. \end{cases}$$

Hence,

$$\text{thirdquartile} = \begin{cases} x_{(3m)} + \frac{3}{4}\left[x_{(3m+1)} - x_{(3m)}\right], & \text{if } n = 4m, \\ x_{(3m)}, & \text{if } n = 4m-1, \\ x_{(3m-1)} + \frac{1}{4}\left[x_{(3m)} - x_{(3m-1)}\right], & \text{if } n = 4m-2, \\ x_{(3m-2)} + \frac{1}{2}\left[x_{(3m-1)} - x_{(3m-2)}\right], & \text{if } n = 4m-3. \end{cases}$$

(4 marks)

UNSEEN

Solutions to Question 2

ILOs addressed: define elementary statistical concepts and terminology such as unbiasedness; analyse and compare statistical properties of simple estimators.

(a) Suppose $\widehat{\theta}$ is an estimator of θ based on a random sample of size n. Define what is meant by the following:

(i) $\widehat{\theta}$ is an unbiased estimator of θ if $E\left(\widehat{\theta}\right) = \theta$; (1 marks)

(ii) the bias of $\widehat{\theta}$ is $E\left(\widehat{\theta}\right) - \theta$; (1 marks)

(iii) the mean squared error of $\widehat{\theta}$ is $E\left[\left(\widehat{\theta} - \theta\right)^2\right]$; (1 marks)

(iv) $\widehat{\theta}$ is a consistent estimator of θ if $\lim_{n\to\infty} E\left[\left(\widehat{\theta} - \theta\right)^2\right] = 0$. (1 marks)

UP TO THIS BOOK WORK.

(b) Suppose $X_1, \ldots, X_n$ are independent Uniform$(0, \theta)$ random variables. Let $\widehat{\theta} = \max\left(X_1, \ldots, X_n\right)$ denote a possible estimator of θ.

(i) Let $Z = \widehat{\theta} = \max\left(X_1, \ldots, X_n\right)$. The cdf of Z is

$$
\begin{aligned}
F_Z(z) &= \Pr\left[\max\left(X_1, \ldots, X_n\right) \leq z\right] \\
&= \Pr\left[X_1 \leq z, \ldots, X_n \leq z\right] \\
&= \Pr\left[X_1 \leq z\right] \cdots \Pr\left[X_n \leq z\right] \\
&= \frac{z}{\theta} \cdots \frac{z}{\theta} \\
&= \frac{z^n}{\theta^n}.
\end{aligned}
$$

The pdf of Z is $f_Z(z) = \frac{nz^{n-1}}{\theta^n}$. Hence,

$$
\begin{aligned}
\text{Bias}(Z) &= E(Z) - \theta \\
&= \int_0^\theta \frac{nz^n}{\theta^n} dz - \theta \\
&= \left[\frac{nz^{n+1}}{\theta^n(n+1)}\right]_0^\theta - \theta \\
&= \frac{n\theta}{n+1} - \theta \\
&= -\frac{\theta}{n+1}.
\end{aligned}
$$

(3 marks)

UNSEEN

(ii) Note that

$$
\begin{aligned}
E\left(Z^2\right) &= \int_0^\theta \frac{nz^{n+1}}{\theta^n} dz \\
&= \left[\frac{nz^{n+2}}{\theta^n(n+2)}\right]_0^\theta \\
&= \frac{n\theta^2}{n+2} - 0 \\
&= \frac{n\theta^2}{n+2},
\end{aligned}
$$

so

$$
\begin{aligned}
\mathrm{MSE}\,(Z) &= \mathrm{Var}(Z) + \left(-\frac{\theta}{n+1}\right)^2 \\
&= E\left(Z^2\right) - [E(Z)]^2 + \left(-\frac{\theta}{n+1}\right)^2 \\
&= \frac{n\theta^2}{n+2} - \left[\frac{n\theta}{n+1}\right]^2 + \left(-\frac{\theta}{n+1}\right)^2.
\end{aligned}
$$

(1 marks)

UNSEEN

(iii) $\widehat{\theta}$ is biased since the bias is not equal to zero. (1 marks)

UNSEEN

(iv) $\widehat{\theta}$ is consistent since the MSE approaches zero. (1 marks)

UNSEEN

Solutions to Question 3

ILOs addressed: define elementary statistical concepts and terminology such as confidence intervals and hypothesis tests.

(a) Suppose we wish to test $H_0 : \mu = \mu_0$ versus $H_0 : \mu \neq \mu_0$.

(i) the Type I error occurs if H_0 is rejected when in fact $\mu = \mu_0$; (1 marks)
SEEN

(ii) the Type II error occurs if H_0 is accepted when in fact $\mu \neq \mu_0$; (1 marks)
SEEN

(iii) the significance level is the probability of type I error. (1 marks)
SEEN

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N\left(\mu, \sigma^2\right)$, where σ is unknown. The rejection region for the following tests are

(i) reject $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ if $\sqrt{n}\left|\overline{X} - \mu_0\right| / S > t_{n-1,1-\frac{\alpha}{2}}$; (1 marks)
SEEN

(ii) reject $H_0 : \mu = \mu_0$ versus $H_1 : \mu < \mu_0$ if $\sqrt{n}\left(\overline{X} - \mu_0\right) / S < t_{n-1,\alpha}$. (1 marks)
SEEN

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N\left(\mu, \sigma^2\right)$, where σ is unknown. Then,

(i) the required probability is

$$
\begin{aligned}
& \Pr\left(\text{Reject } H_0 \mid H_1 \text{ is true}\right) \\
= \; & \Pr\left(\frac{\sqrt{n}\left|\overline{X}-\mu_0\right|}{S} > t_{n-1,1-\frac{\alpha}{2}} \mid \mu \neq \mu_0\right) \\
= \; & \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu_0\right)}{S} > t_{n-1,1-\frac{\alpha}{2}} \text{ or } \frac{\sqrt{n}\left(\overline{X}-\mu_0\right)}{S} < -t_{n-1,1-\frac{\alpha}{2}} \mid \mu \neq \mu_0\right) \\
= \; & \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu_0\right)}{S} > t_{n-1,1-\frac{\alpha}{2}} \mid \mu \neq \mu_0\right) + \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu_0\right)}{S} < -t_{n-1,1-\frac{\alpha}{2}} \mid \mu \neq \mu_0\right) \\
= \; & \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu+\mu-\mu_0\right)}{S} > t_{n-1,1-\frac{\alpha}{2}} \mid \mu \neq \mu_0\right) + \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu+\mu-\mu_0\right)}{S} < -t_{n-1,1-\frac{\alpha}{2}} \mid \mu \right. \\
= \; & \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu\right)}{S} > t_{n-1,1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{S} \mid \mu \neq \mu_0\right) + \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu\right)}{S} < -t_{n-1,1-\frac{\alpha}{2}} - \sqrt{n} \right. \\
= \; & \Pr\left(T_{n-1} > t_{n-1,1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{S}\right) + \Pr\left(T_{n-1} < -t_{n-1,1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{S}\right) \\
= \; & 1 - \Pr\left(T_{n-1} \leq t_{n-1,1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{S}\right) + \Pr\left(T_{n-1} < -t_{n-1,1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{S}\right) \\
= \; & 1 - F_{T_{n-1}}\left(t_{n-1,1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{S}\right) + F_{T_{n-1}}\left(-t_{n-1,1-\frac{\alpha}{2}} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{S}\right).
\end{aligned}
$$

(3 marks)

UNSEEN

(ii) the required probability is

$$
\begin{aligned}
& \Pr\left(\text{Reject } H_0 \mid H_1 \text{ is true}\right) \\
= \; & \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu_0\right)}{S} < t_{n-1,\alpha} \mid \mu < \mu_0\right) \\
= \; & \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu+\mu-\mu_0\right)}{S} < t_{n-1,\alpha} \mid \mu < \mu_0\right) \\
= \; & \Pr\left(\frac{\sqrt{n}\left(\overline{X}-\mu\right)}{S} < t_{n-1,\alpha} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{S} \mid \mu < \mu_0\right) \\
= \; & \Pr\left(T_{n-1} < t_{n-1,\alpha} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{S}\right) \\
= \; & F_{T_{n-1}}\left(t_{n-1,\alpha} - \frac{\sqrt{n}\left(\mu-\mu_0\right)}{S}\right).
\end{aligned}
$$

(2 marks)

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Solutions to Question 4

ILOs addressed: define elementary statistical concepts and terminology such as confidence intervals and hypothesis tests; conduct statistical inferences, including confidence intervals and hypothesis tests, in simple one and two-sample situations; sampling distributions.

(a) Let $\mathbf{X} = (X_1, \ldots, X_n)$, with $X_1, \ldots, X_n$ an independent random sample from a distribution F_X with unknown parameter θ. Let $I(\mathbf{X}) = [a(\mathbf{X}), b(\mathbf{X})]$ denote an interval estimator for θ.

(i) $I(\mathbf{X})$ is a $100(1-\alpha)\%$ confidence interval if

$$\Pr\left(a(\mathbf{X}) < \theta < b(\mathbf{X})\right) = 1 - \alpha;$$

(1 marks)

SEEN

(ii) the coverage probability of $I(\mathbf{X})$ is

$$\Pr\left(a(\mathbf{X}) < \theta < b(\mathbf{X})\right);$$

(1 marks)

SEEN

(iii) the coverage length of $I(\mathbf{X})$ is $b(\mathbf{X}) - a(\mathbf{X})$. (1 marks)

SEEN

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$.

(i) if σ is known then $\sqrt{n}\left(\overline{X} - \mu\right)/\sigma \sim N(0,1)$. So,

$$\begin{aligned}
& \Pr\left(z_{\alpha/2} < \frac{\sqrt{n}\left(\overline{X} - \mu\right)}{\sigma} < z_{1-\alpha/2}\right) = 1 - \alpha \\
\Leftrightarrow\ & \Pr\left(\frac{\sigma}{\sqrt{n}} z_{\alpha/2} < \overline{X} - \mu < \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}\right) = 1 - \alpha \\
\Leftrightarrow\ & \Pr\left(-\overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} < -\mu < -\overline{X} + \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}\right) = 1 - \alpha \\
\Leftrightarrow\ & \Pr\left(\overline{X} - \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2} < \mu < \overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right) = 1 - \alpha.
\end{aligned}$$

Hence, a $100(1-\alpha)\%$ confidence interval for μ is

$$\left(\overline{X} - \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}, \overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right).$$

(1 marks)

SEEN

(ii) if σ is not known then $\sqrt{n}\left(\overline{X}-\mu\right)/S \sim t_{n-1}$. So,

$$
\begin{aligned}
& \Pr\left(t_{n-1,\alpha/2} < \frac{\sqrt{n}\left(\overline{X}-\mu\right)}{S} < t_{n-1,1-\alpha/2}\right) = 1-\alpha \\
\Leftrightarrow\ & \Pr\left(\frac{S}{\sqrt{n}} t_{n-1,\alpha/2} < \overline{X}-\mu < \frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2}\right) = 1-\alpha \\
\Leftrightarrow\ & \Pr\left(-\overline{X}+\frac{S}{\sqrt{n}} t_{n-1,\alpha/2} < -\mu < -\overline{X}+\frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2}\right) = 1-\alpha \\
\Leftrightarrow\ & \Pr\left(\overline{X}-\frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2} < \mu < \overline{X}-\frac{S}{\sqrt{n}} t_{n-1,\alpha/2}\right) = 1-\alpha.
\end{aligned}
$$

Hence, a $100(1-\alpha)\%$ confidence interval for μ is

$$
\left(\overline{X}-\frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2}, \overline{X}-\frac{S}{\sqrt{n}} t_{n-1,\alpha/2}\right).
$$

(1 marks)

SEEN

(c) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from $Uniform\,[0,a]$.

(i) The cumulative distribution function $\min\left(X_1, X_2, \ldots, X_n\right) = Z$ say, is

$$
\begin{aligned}
F_Z(z) &= \Pr\left(Z \leq z\right) \\
&= \Pr\left(\min\left(X_1, X_2, \ldots, X_n\right) \leq z\right) \\
&= 1-\Pr\left(\min\left(X_1, X_2, \ldots, X_n\right) > z\right) \\
&= 1-\Pr\left(X_1 > z, \ldots, X_n > z\right) \\
&= 1-\Pr\left(X_1 > z\right)\cdots\Pr\left(X_n > z\right) \\
&= 1-\left[1-\Pr\left(X_1 \leq z\right)\right]\cdots\left[1-\Pr\left(X_n \leq z\right)\right] \\
&= 1-\left[1-\frac{z}{a}\right]\cdots\left[1-\frac{z}{a}\right] \\
&= 1-\left[1-\frac{z}{a}\right]^n
\end{aligned}
$$

for $0 < z < a$. (2 marks)

UNSEEN

(ii) The $\left(\frac{\alpha}{2}\right)$th and $\left(1-\frac{\alpha}{2}\right)$th percentiles of Z are $a\left[1-\left(1-\frac{\alpha}{2}\right)^{1/n}\right]$ and $a\left[1-\left(\frac{\alpha}{2}\right)^{1/n}\right]$, respectively. So,

$$
\Pr\left(a\left[1-\left(1-\frac{\alpha}{2}\right)^{1/n}\right] \leq Z \leq a\left[1-\left(\frac{\alpha}{2}\right)^{1/n}\right]\right) = 1-\alpha,
$$

which can be rewritten as

$$\Pr\left(Z\left[1-\left(\frac{\alpha}{2}\right)^{-1/n}\right] \leq a \leq Z\left[1-\left(1-\frac{\alpha}{2}\right)^{-1/n}\right]\right) = 1-\alpha.$$

Hence, a $100(1-\alpha)\%$ confidence interval for a is

$$\left[Z\left[1-\left(\frac{\alpha}{2}\right)^{-1/n}\right], Z\left[1-\left(1-\frac{\alpha}{2}\right)^{-1/n}\right]\right].$$

(3 marks)

UNSEEN

Solutions to Question 5

ILOs addressed: analyse and compare statistical properties of simple estimators.

Suppose $X \sim \text{Binomial}(m, p)$ and $Y \sim \text{Binomial}(n, p)$ are independent random variables. Consider the following estimators for p:

$$\widehat{p_1} = \frac{X}{2m} + \frac{Y}{2n}$$

and

$$\widehat{p_2} = \frac{X + Y}{m + n}.$$

(i) The bias of the first estimator is

$$\begin{aligned}
\text{Bias}\left(\widehat{p_1}\right) &= E\left(\widehat{p_1}\right) - p \\
&= E\left[\frac{1}{2}\left(\frac{X}{m} + \frac{Y}{n}\right)\right] - p \\
&= \frac{1}{2}\left[\frac{E(X)}{m} + \frac{E(Y)}{n}\right] - p \\
&= \frac{1}{2}\left(\frac{mp}{m} + \frac{np}{n}\right) - p \\
&= \frac{1}{2}(p + p) - p \\
&= 0.
\end{aligned}$$

(2 marks)

UNSEEN

(ii) The bias of the second estimator is

$$\begin{aligned}
\text{Bias}\left(\widehat{p_2}\right) &= E\left(\widehat{p_2}\right) - p \\
&= E\left(\frac{X + Y}{m + n}\right) - p \\
&= \frac{E(X + Y)}{m + n} - p \\
&= \frac{E(X) + E(Y)}{m + n} - p \\
&= \frac{mp + np}{m + n} - p \\
&= p - p \\
&= 0.
\end{aligned}$$

(2 marks)

UNSEEN

(iii) The mean squared error of the first estimator is

$$\begin{aligned}
\text{MSE}\left(\widehat{p_1}\right) &= \text{Var}\left(\widehat{p_1}\right) \\
&= \text{Var}\left(\frac{1}{2}\left(\frac{X}{m}+\frac{Y}{n}\right)\right) \\
&= \frac{1}{4}\text{Var}\left(\frac{X}{m}+\frac{Y}{n}\right) \\
&= \frac{1}{4}\left[\frac{\text{Var}(X)}{m^2}+\frac{\text{Var}(Y)}{n^2}\right] \\
&= \frac{1}{4}\left[\frac{mp(1-p)}{m^2}+\frac{np(1-p)}{n^2}\right] \\
&= \frac{1}{4}\left[\frac{p(1-p)}{m}+\frac{p(1-p)}{n}\right] \\
&= \frac{p(1-p)}{4}\left(\frac{1}{m}+\frac{1}{n}\right).
\end{aligned}$$

(2 marks)

UNSEEN

(iv) The mean squared error of the second estimator is

$$\begin{aligned}
\text{MSE}\left(\widehat{p_2}\right) &= \text{Var}\left(\widehat{p_2}\right) \\
&= \text{Var}\left(\frac{X+Y}{m+n}\right) \\
&= \frac{1}{(m+n)^2}\text{Var}\left(X+Y\right) \\
&= \frac{1}{(m+n)^2}\left[\text{Var}(X)+\text{Var}(Y)\right] \\
&= \frac{1}{(m+n)^2}\left[mp(1-p)+np(1-p)\right] \\
&= \frac{p(1-p)}{m+n}.
\end{aligned}$$

(2 marks)

UNSEEN

(v) Both estimators have zero bias, so they are equally good. (2 marks)

UNSEEN

(vi) $\widehat{p_2}$ is the better since it has smaller MSE than $\widehat{p_1}$ since

$$
\begin{aligned}
& \frac{p(1-p)}{m+n} \leq \frac{p(1-p)}{4}\left(\frac{1}{m}+\frac{1}{n}\right) \\
\iff\ & \frac{1}{m+n} \leq \frac{1}{4}\left(\frac{1}{m}+\frac{1}{n}\right) \\
\iff\ & \frac{1}{m+n} \leq \frac{1}{4}\frac{m+n}{mn} \\
\iff\ & 4mn \leq (m+n)^2 \\
\iff\ & 4mn \leq m^2+n^2+2mn \\
\iff\ & 0 \leq m^2+n^2-2mn \\
\iff\ & 0 \leq (m-n)^2.
\end{aligned}
$$

(2 marks)

UNSEEN

Solutions to Question 6

ILOs addressed: analyse statistical properties of simple estimators.

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function $\frac{x^2}{\sigma^3} \exp\left(-\frac{x^3}{3\sigma^3}\right)$ for $x > 0$.

(i) The likelihood function of σ^2 is

$$
\begin{aligned}
L\left(\sigma^2\right) &= \prod_{i=1}^{n} \left[\frac{X_i^2}{\sigma^3} \exp\left(-\frac{X_i^3}{3\sigma^3}\right) \right] \\
&= \frac{1}{\sigma^{3n}} \left(\prod_{i=1}^{n} X_i^2 \right) \exp\left(-\frac{1}{3\sigma^3} \sum_{i=1}^{n} X_i^3 \right).
\end{aligned}
$$

(4 marks)

UNSEEN

(ii) The log likelihood function of σ is

$$
\log L(\sigma) = -3n \log \sigma + 2 \prod_{i=1}^{n} \log X_i - \frac{1}{3\sigma^3} \sum_{i=1}^{n} X_i^3.
$$

The derivative with respect to σ is

$$
\frac{d \log L(\sigma)}{d\sigma} = -\frac{3n}{\sigma} + \frac{1}{\sigma^4} \sum_{i=1}^{n} X_i^3.
$$

Setting this to zero gives

$$
\widehat{\sigma^3} = \frac{1}{3n} \sum_{i=1}^{n} X_i^3.
$$

This is a maximum likelihood estimator since

$$
\begin{aligned}
\frac{d^2 \log L(\sigma)}{d\sigma^2} &= \frac{3n}{\sigma^2} - \frac{4}{\sigma^5} \sum_{i=1}^{n} X_i^3 \\
&= \frac{1}{\sigma^5} \left[3n\sigma^3 - 4 \sum_{i=1}^{n} X_i^3 \right] \\
&= \frac{1}{\sigma^5} \left[3n \frac{1}{3n} \sum_{i=1}^{n} X_i^3 - 4 \sum_{i=1}^{n} X_i^2 \right] \\
&< 0
\end{aligned}
$$

at $\sigma = \widehat{\sigma}$. (4 marks)

UNSEEN

(iii) By the invariance principle, the maximum likelihood estimator of σ is

$$\widehat{\sigma} = \left[\frac{1}{3n} \sum_{i=1}^{n} X_i^3 \right]^{1/3}$$

(4 marks)

UNSEEN

(iv) The bias of $\widehat{\sigma^3}$ is

$$\begin{aligned}
\text{Bias}\left(\widehat{\sigma^3}\right) &= E\left(\widehat{\sigma^3}\right) - \sigma^3 \\
&= E\left(\frac{1}{3n} \sum_{i=1}^{n} X_i^3\right) - \sigma^3 \\
&= \frac{1}{3n} \sum_{i=1}^{n} E\left(X_i^3\right) - \sigma^3 \\
&= \frac{1}{3n\sigma^3} \sum_{i=1}^{n} \int_0^{\infty} x^5 \exp\left(-\frac{x^3}{3\sigma^3}\right) dx - \sigma^3 \\
&= \frac{\sigma^3}{n} \sum_{i=1}^{n} \int_0^{\infty} y \exp\left(-y\right) dy - \sigma^3 \\
&= \frac{\sigma^3}{n} \sum_{i=1}^{n} \Gamma(2) - \sigma^3 \\
&= \frac{\sigma^3}{n} \sum_{i=1}^{n} 1 - \sigma^3 \\
&= 0.
\end{aligned}$$

Hence, $\widehat{\sigma^3}$ is unbiased for σ^3. (4 marks)

UNSEEN

(v) The mean squared error of $\widehat{\sigma^3}$ is

$$\begin{aligned}
\text{MSE}\left(\widehat{\sigma^3}\right) &= \text{Var}\left(\widehat{\sigma^3}\right) \\
&= \text{Var}\left(\frac{1}{3n}\sum_{i=1}^{n} X_i^3\right) \\
&= \frac{1}{9n^2}\sum_{i=1}^{n} \text{Var}\left(X_i^3\right) \\
&= \frac{1}{9n^2}\sum_{i=1}^{n} \left\{E\left(X_i^6\right) - \left[E\left(X_i^3\right)\right]^2\right\} \\
&= \frac{1}{9n^2}\sum_{i=1}^{n} \left\{E\left(X_i^6\right) - \left[3\sigma^3\right]^2\right\} \\
&= \frac{1}{9n^2}\sum_{i=1}^{n} \left\{9\sigma^6 \int_0^\infty y^2 \exp\left(-y\right) dy - 9\sigma^6\right\} \\
&= \frac{1}{9n^2}\sum_{i=1}^{n} \left\{9\sigma^6\Gamma(3) - 9\sigma^6\right\} \\
&= \frac{1}{9n^2}\sum_{i=1}^{n} \left\{18\sigma^6 - 9\sigma^6\right\} \\
&= \frac{\sigma^6}{n}.
\end{aligned}$$

Hence, $\widehat{\sigma^3}$ is consistent for σ^3. (4 marks)

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Solutions to Question 7

ILOs addressed: analyse statistical properties of simple estimators.

An electrical circuit consists of four batteries connected in series to a lightbulb. We model the battery lifetimes X_1, X_2, X_3 as independent and identically distributed $Uni(0, \theta)$ random variables. Our experiment to measure the operating time of the circuit is stopped when any one of the batteries fails. Hence, the only random variable we observe is $Y = \min(X_1, X_2, X_3)$.

(i) The cdf of Y is

$$\begin{aligned}\Pr(Y \leq y) &= 1 - \Pr\left[\min(X_1, X_2, X_3) > y\right] \\ &= 1 - \Pr(X_1 > y)\Pr(X_2 > y)\Pr(X_3 > y) \\ &= 1 - \Pr^3(X > y) \\ &= 1 - (1 - y/\theta)^3.\end{aligned}$$

(4 marks)

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(ii) The likelihood function of θ is

$$L(\theta) = 3(\theta - y)^2/\theta^3$$

for $0 < y < \theta$. (4 marks)

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(iii) The log-likelihood function is

$$\log L(\theta) = \log 3 + 2\log(\theta - y) - 3\log\theta$$

and

$$\frac{d \log L(\theta)}{d\theta} = \frac{2}{\theta - y} - \frac{3}{\theta}.$$

Setting $d \log L(\theta)/d\theta = 0$ gives $\widehat{\theta} = 3y$. This is an MLE since

$$\frac{d^2 \log L(\theta)}{d\theta^2} = -\frac{2}{(\theta - y)^2} + \frac{3}{\theta^2}$$

at $\widehat{\theta} = 3y$ is negative. (4 marks)

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(iv) The bias of $\widehat{\theta}$ is

$$\begin{aligned} Bias\left(\widehat{\theta}\right) &= E\left(\widehat{\theta}\right) - \theta \\ &= 9\theta^{-3}\int_0^\theta y(\theta-y)^2 dy - \theta \\ &= 9\theta\int_0^1 y(1-y)^2 dy - \theta \\ &= \frac{3\theta}{4} - \theta \\ &= -\frac{\theta}{4}, \end{aligned}$$

so the estimator is biased. (4 marks)

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(v) The variance of $\widehat{\theta}$ is

$$\begin{aligned} Var\left(\widehat{\theta}\right) &= E\left(\widehat{\theta}^2\right) - E^2\left(\widehat{\theta}\right) \\ &= 27\theta^{-3}\int_0^\theta y^2(\theta-y)^2 dy - \frac{9\theta^2}{16} \\ &= 27\theta^2\int_0^1 y^2(1-y)^2 dy - \frac{9\theta^2}{16} \\ &= \frac{9\theta^2}{10} - \frac{9\theta^2}{16} \\ &= \frac{27\theta^2}{80}. \end{aligned}$$

So, the mean squared error of $\widehat{\lambda}$ is

$$MSE\left(\widehat{\theta}\right) = Var\left(\widehat{\theta}\right) + Bias^2\left(\widehat{\theta}\right) = \frac{27\theta^2}{80} + \frac{\theta^2}{8} = \frac{37\theta^2}{80}.$$

(4 marks)

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