MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 8

Suppose X_i distributed as N $(ia, 1)$, $i = 1, \ldots, n$ are independent random variables. The likelihood function of a is

$$L(a) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x_i - ia)^2}{2}\right] = \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i - ia)^2\right].$$

The log-likelihood function is

$$\log L(a) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i - ia)^2.$$

The derivative with respect to a is

$$\begin{aligned}
\frac{d\log L(a)}{da} &= \sum_{i=1}^{n} i\,(x_i - ia) \\
&= \left(\sum_{i=1}^{n} i x_i\right) - a\sum_{i=1}^{n} i^2 \\
&= \left(\sum_{i=1}^{n} i x_i\right) - a\frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

Setting this to zero and solving for a, we obtain

$$\widehat{a} = \frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n}(ix_i).$$

This is a maximum likelihood estimator since

$$\frac{d^2\log L(a)}{da^2} = -\frac{n(n+1)(2n+1)}{6} < 0.$$

So, the correct answer is a).