MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 7

Suppose $X_i \sim N\left(0, i\sigma^2\right)$, $i = 1, \ldots, n$ are independent random variables. The likelihood function of σ^2 is

$$L\left(\sigma^2\right) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sqrt{i}\sigma} \exp\left(-\frac{x_i^2}{2i\sigma^2}\right) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{n/2}\sigma^n} \left(\prod_{i=1}^{n} \frac{1}{\sqrt{i}}\right) \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} \frac{x_i^2}{i}\right).$$

The log-likelihood function is

$$\log L\left(\sigma^2\right) = -\frac{n}{2}\log(2\pi) - n\log\sigma + \log\left(\prod_{i=1}^{n} \frac{1}{\sqrt{i}}\right) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \frac{x_i^2}{i}.$$

The derivative with respect to σ is

$$\frac{d \log L\left(\sigma^2\right)}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} \frac{x_i^2}{i}.$$

Setting this to zero and solving for σ^2, we obtain

$$\widehat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i^2}{i}.$$

This is a maximum likelihood estimator since

$$\begin{aligned}
\frac{d^2 \log L\left(\sigma^2\right)}{d\sigma^2} &= \frac{n}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=1}^{n} \frac{x_i^2}{i} \\
&= \frac{1}{\sigma^4}\left[n\sigma^2 - 3\sum_{i=1}^{n} \frac{x_i^2}{i}\right] \\
&= \frac{1}{\sigma^4}\left[\sum_{i=1}^{n} \frac{x_i^2}{i} - 3\sum_{i=1}^{n} \frac{x_i^2}{i}\right] \\
&= -\frac{2}{\sigma^4} \sum_{i=1}^{n} \frac{x_i^2}{i}
\end{aligned}$$

when $\sigma^2 = \widehat{\sigma^2}$.

So, the correct answer is a).