MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 6

Suppose $X_1, \ldots, X_n$ is a random sample from Uniform$[0, a]$. Suppose $\widehat{a} = \max(X_1, \ldots, X_n)$ is an estimator of a. Let $Z = \max(X_1, \ldots, X_n)$. The cumulative distribution function of Z is

$$\begin{aligned} F_Z(z) &= \Pr(Z \leq z) \\ &= \Pr\left(\max(X_1, \ldots, X_n) \leq z\right) \\ &= \Pr\left(X_1 \leq z, \ldots, X_n \leq z\right) \\ &= \Pr\left(X_1 \leq z\right) \cdots \Pr\left(X_n \leq z\right) \\ &= \left[\Pr\left(X \leq z\right)\right]^n \\ &= \left[\frac{z}{a}\right]^n. \end{aligned}$$

So, the probability density function of Z is

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{n z^{n-1}}{a^n}.$$

The bias is

$$\begin{aligned} Bias(Z) &= E(Z) - a \\ &= \int_a^1 z \frac{n z^{n-1}}{a^n} dz - a \\ &= \frac{n}{a^n} \int_0^a z^n dz - a \\ &= \frac{n}{a^n} \left[\frac{z^{n+1}}{n+1}\right]_0^a - a \\ &= \frac{n}{a^n} \left[\frac{a^{n+1}}{n+1} - 0\right] - a \\ &= \frac{na}{n+1} - a \\ &= -\frac{a}{n+1}. \end{aligned}$$

The mean squared error is

$$\begin{aligned} MSE(Z) &= Var(Z) + \left[-\frac{a}{n+1}\right]^2 \\ &= E\left(Z^2\right) - [E(Z)]^2 + \left[\frac{a}{n+1}\right]^2 \end{aligned}$$

$$
\begin{aligned}
&= E\left(Z^2\right) - \left[\frac{na}{n+1}\right]^2 + \left[\frac{a}{n+1}\right]^2 \\
&= \int_0^a z^2 \frac{nz^{n-1}}{a^n} dz - \left[\frac{na}{n+1}\right]^2 + \left[\frac{a}{n+1}\right]^2 \\
&= \frac{n}{a^n} \int_0^a z^{n+1} dz - \left[\frac{na}{n+1}\right]^2 + \left[\frac{a}{n+1}\right]^2 \\
&= \frac{n}{a^n} \left[\frac{z^{n+2}}{n+2}\right]_0^a - \left[\frac{na}{n+1}\right]^2 + \left[\frac{a}{n+1}\right]^2 \\
&= \frac{n}{a^n} \left[\frac{a^{n+2}}{n+2} - 0\right] - \left[\frac{na}{n+1}\right]^2 + \left[\frac{a}{n+1}\right]^2 \\
&= \frac{na^2}{n+2} - \left[\frac{na}{n+1}\right]^2 + \left[\frac{a}{n+1}\right]^2 \\
&= \frac{2a^2}{(n+1)(n+2)}.
\end{aligned}
$$

Hence, the estimator is consistent since

$$
\lim_{n\to\infty} MSE(Z) = \lim_{n\to\infty} \frac{2a^2}{(n+1)(n+2)} = 0.
$$

So, the correct answer is c).