MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 5

Suppose $X_1, \ldots, X_n$ is a random sample from Uniform$[0, a]$. Suppose $\widehat{a} = \max(X_1, \ldots, X_n)$ is an estimator of a. Let $Z = \max(X_1, \ldots, X_n)$. The cumulative distribution function of Z is

$$\begin{aligned} F_Z(z) &= \Pr(Z \leq z) \\ &= \Pr(\max(X_1, \ldots, X_n) \leq z) \\ &= \Pr(X_1 \leq z, \ldots, X_n \leq z) \\ &= \Pr(X_1 \leq z) \cdots \Pr(X_n \leq z) \\ &= [\Pr(X \leq z)]^n \\ &= \left[\frac{z}{a}\right]^n. \end{aligned}$$

So, the probability density function of Z is

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{n z^{n-1}}{a^n}.$$

The bias is

$$\begin{aligned} Bias(Z) &= E(Z) - a \\ &= \int_a^1 z \frac{n z^{n-1}}{a^n} dz - a \\ &= \frac{n}{a^n} \int_0^a z^n dz - a \\ &= \frac{n}{a^n} \left[\frac{z^{n+1}}{n+1}\right]_0^a - a \\ &= \frac{n}{a^n} \left[\frac{a^{n+1}}{n+1} - 0\right] - a \\ &= \frac{na}{n+1} - a \\ &= -\frac{a}{n+1}. \end{aligned}$$

Hence, the estimator is not unbiased.

So, the correct answer is b).