MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 2

Suppose that $X_1, \ldots, X_n$ is a random sample from a distribution specified by the probability density function

$$f_X(x) = 0.5 \exp\left(- \mid x \mid\right)$$

for $-\infty < x < \infty$. From the lecture notes, the mean and variance of the sampling distribution of $\overline{X}$ are

$$E\left(\overline{X}\right) = \mu$$

and

$$Var\left(\overline{X}\right) = \frac{\sigma^2}{n},$$

where $\mu = E(X)$ and $\sigma^2 = Var(X)$. From the given probability density function,

$$\begin{aligned}
\mu &= 0.5 \int_{-\infty}^{\infty} x \exp\left(- \mid x \mid\right) dx \\
&= 0.5 \left[\int_{-\infty}^{0} x \exp\left(- \mid x \mid\right) dx + \int_{0}^{\infty} x \exp\left(- \mid x \mid\right) dx\right] \\
&= 0.5 \left[\int_{-\infty}^{0} x \exp\left(x\right) dx + \int_{0}^{\infty} x \exp\left(-x\right) dx\right] \\
&= 0.5 \left[\int_{\infty}^{0} x \exp\left(-x\right) dx + \int_{0}^{\infty} x \exp\left(-x\right) dx\right] \\
&= 0.5 \left[-\int_{0}^{\infty} x \exp\left(-x\right) dx + \int_{0}^{\infty} x \exp\left(-x\right) dx\right] \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\sigma^2 &= E\left(X^2\right) - \left[E(X)\right]^2 \\
&= 0.5 \int_{-\infty}^{\infty} x^2 \exp\left(- \mid x \mid\right) dx - 0 \\
&= 0.5 \int_{-\infty}^{\infty} x^2 \exp\left(- \mid x \mid\right) dx \\
&= 0.5 \left[\int_{-\infty}^{0} x^2 \exp\left(- \mid x \mid\right) dx + \int_{0}^{\infty} x^2 \exp\left(- \mid x \mid\right) dx\right] \\
&= 0.5 \left[\int_{-\infty}^{0} x^2 \exp\left(x\right) dx + \int_{0}^{\infty} x^2 \exp\left(-x\right) dx\right]
\end{aligned}$$

$$
\begin{aligned}
&= 0.5\left[-\int_{\infty}^{0} x^2 \exp(-x)\, dx + \int_{0}^{\infty} x^2 \exp(-x)\, dx\right] \\
&= 0.5\left[\int_{0}^{\infty} x^2 \exp(-x)\, dx + \int_{0}^{\infty} x^2 \exp(-x)\, dx\right] \\
&= \int_{0}^{\infty} x^2 \exp(-x)\, dx \\
&= \Gamma(3) \\
&= 2! \\
&= 2.
\end{aligned}
$$

Hence,

$$E\left(\overline{X}\right) = 0$$

and

$$Var\left(\overline{X}\right) = \frac{2}{n}.$$

So, the correct answer is b).