MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 8

Suppose $X_1, \ldots, X_n$ is a random sample from a distribution specified by the cumulative distribution function $F(x) = 1 - (1-x)^a$ for $a > 0$ and $0 < x < 1$. The corresponding probability density function is $f(x) = a(1-x)^{a-1}$ for $a > 0$ and $0 < x < 1$. The likelihood function of a is

$$L(a) = \prod_{i=1}^{n} \left[a\,(1-x_i)^{a-1} \right] = a^n \left[\prod_{i=1}^{n} (1-x_i) \right]^{a-1}.$$

The log-likelihood function is

$$\log L(a) = n \log a + (a-1) \log \prod_{i=1}^{n} (1-x_i) = n \log a + (a-1) \sum_{i=1}^{n} \log (1-x_i).$$

The derivative with respect to a is

$$\frac{d \log L(a)}{da} = \frac{n}{a} + \sum_{i=1}^{n} \log (1-x_i).$$

Setting this to zero and solving for a, we obtain

$$\widehat{a} = -\frac{n}{\sum_{i=1}^{n} \log (1-x_i)}.$$

This is a maximum likelihood estimator since

$$\frac{d^2 \log L(a)}{da^2} = -\frac{n}{a^2} < 0.$$

So, the correct answer is c).