MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 6

Suppose $X_1, \ldots, X_n$ is a random sample from Uniform$[a, 1]$. Suppose $\widehat{a} = \min(X_1, \ldots, X_n)$ is an estimator of a. Let $Z = \min(X_1, \ldots, X_n)$. The cumulative distribution function of Z is

$$\begin{aligned}
F_Z(z) &= \Pr(Z \leq z) \\
&= 1 - \Pr(Z > z) \\
&= 1 - \Pr(\min(X_1, \ldots, X_n) > z) \\
&= 1 - \Pr(X_1 > z, \ldots, X_n > z) \\
&= 1 - \Pr(X_1 > z) \cdots \Pr(X_n > z) \\
&= 1 - [\Pr(X > z)]^n \\
&= 1 - [1 - \Pr(X \leq z)]^n \\
&= 1 - \left[1 - \frac{z-a}{1-a}\right]^n \\
&= 1 - \left[\frac{1-z}{1-a}\right]^n.
\end{aligned}$$

So, the probability density function of Z is

$$f_Z(z) = \frac{d}{dz} F_Z(z) = n\frac{(1-z)^{n-1}}{(1-a)^n}.$$

We know from quiz 5 that

$$E(Z) = \left[1 - \frac{n(1-a)}{n+1}\right]$$

and

$$Bias(Z) = \frac{1-a}{n+1}.$$

The mean squared error is

$$\begin{aligned}
MSE(Z) &= Var(Z) + \left[\frac{1-a}{n+1}\right]^2 \\
&= E\left(Z^2\right) - [E(Z)]^2 + \left[\frac{1-a}{n+1}\right]^2 \\
&= E\left(Z^2\right) - \left[1 - \frac{n(1-a)}{n+1}\right]^2 + \left[\frac{1-a}{n+1}\right]^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n}{(1-a)^n}\int_a^1 z^2(1-z)^{n-1}dz - \left[1-\frac{n(1-a)}{n+1}\right]^2 + \left[\frac{1-a}{n+1}\right]^2 \\
&= \frac{n}{(1-a)^n}\int_a^1 \left[1-(1-z)\right]^2(1-z)^{n-1}dz - \left[1-\frac{n(1-a)}{n+1}\right]^2 + \left[\frac{1-a}{n+1}\right]^2 \\
&= \frac{n}{(1-a)^n}\int_a^1 \left[1-2(1-z)+(1-z)^2\right](1-z)^{n-1}dz - \left[1-\frac{n(1-a)}{n+1}\right]^2 + \left[\frac{1-a}{n+1}\right]^2 \\
&= \frac{n}{(1-a)^n}\int_a^1 \left[(1-z)^{n-1}-2(1-z)^n+(1-z)^{n+1}\right]dz - \left[1-\frac{n(1-a)}{n+1}\right]^2 + \left[\frac{1-a}{n+1}\right]^2 \\
&= \frac{n}{(1-a)^n}\left[-\frac{(1-z)^n}{n}+\frac{2(1-z)^{n+1}}{n+1}-\frac{(1-z)^{n+2}}{n+2}\right]_a^1 - \left[1-\frac{n(1-a)}{n+1}\right]^2 + \left[\frac{1-a}{n+1}\right]^2 \\
&= \frac{n}{(1-a)^n}\left[\frac{(1-a)^n}{n}-\frac{2(1-a)^{n+1}}{n+1}+\frac{(1-a)^{n+2}}{n+2}\right] - \left[1-\frac{n(1-a)}{n+1}\right]^2 + \left[\frac{1-a}{n+1}\right]^2 \\
&= \left[1-\frac{2(1-a)n}{n+1}+\frac{(1-a)^2n}{n+2}\right] - \left[1-\frac{n(1-a)}{n+1}\right]^2 + \left[\frac{1-a}{n+1}\right]^2 \\
&= \frac{(1-a)^2n}{n+2} - \frac{n^2(1-a)^2}{(n+1)^2} + \left[\frac{1-a}{n+1}\right]^2 \\
&= \frac{2(1-a)^2}{(n+1)(n+2)}.
\end{aligned}
$$

Hence, the estimator is consistent since

$$\lim_{n\to\infty} MSE(Z) = \lim_{n\to\infty}\frac{2(1-a)^2}{(n+1)(n+2)} = 0.$$

So, the correct answer is d).