MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 5

Suppose $X_1, \ldots, X_n$ is a random sample from Uniform$[a, 1]$. Suppose $\widehat{a} = \min(X_1, \ldots, X_n)$ is an estimator of a. Let $Z = \min(X_1, \ldots, X_n)$. The cumulative distribution function of Z is

$$\begin{aligned}
F_Z(z) &= \Pr(Z \leq z) \\
&= 1 - \Pr(Z > z) \\
&= 1 - \Pr(\min(X_1, \ldots, X_n) > z) \\
&= 1 - \Pr(X_1 > z, \ldots, X_n > z) \\
&= 1 - \Pr(X_1 > z) \cdots \Pr(X_n > z) \\
&= 1 - [\Pr(X > z)]^n \\
&= 1 - [1 - \Pr(X \leq z)]^n \\
&= 1 - \left[1 - \frac{z-a}{1-a}\right]^n \\
&= 1 - \left[\frac{1-z}{1-a}\right]^n .
\end{aligned}$$

So, the probability density function of Z is

$$f_Z(z) = \frac{d}{dz} F_Z(z) = n \frac{(1-z)^{n-1}}{(1-a)^n}.$$

The bias is

$$\begin{aligned}
Bias(Z) &= E(Z) - a \\
&= \frac{n}{(1-a)^n} \int_a^1 z(1-z)^{n-1} dz - a \\
&= \frac{n}{(1-a)^n} \int_a^1 [1 - (1-z)] (1-z)^{n-1} dz - a \\
&= \frac{n}{(1-a)^n} \int_a^1 \left[(1-z)^{n-1} - (1-z)^n\right] dz - a \\
&= \frac{n}{(1-a)^n} \left[-\frac{(1-z)^n}{n} + \frac{(1-z)^{n+1}}{n+1}\right]_a^1 - a \\
&= \frac{n}{(1-a)^n} \left[\frac{(1-a)^n}{n} - \frac{(1-a)^{n+1}}{n+1}\right] - a \\
&= \left[1 - \frac{n(1-a)}{n+1}\right] - a \\
&= \frac{1-a}{n+1}.
\end{aligned}$$

Hence, the estimator is not unbiased.

So, the correct answer is a).