MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 8

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi} x\theta} \exp\left[-\frac{(\log x - \theta)^2}{2\theta^2}\right]$$

for $x > 0$ and $\theta > 0$.

The likelihood function of θ is

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{2\pi} X_i \theta} \exp\left[-\frac{(\log X_i - \theta)^2}{2\theta^2}\right] \right\} \\ &= \frac{1}{(2\pi)^{n/2}\theta^n} \left(\prod_{i=1}^{n} X_i\right)^{-1} \exp\left[-\frac{1}{2\theta^2}\sum_{i=1}^{n} (\log X_i - \theta)^2\right]. \end{aligned}$$

Its log is

$$\begin{aligned} \log L(\theta) &= -\frac{n}{2}\log(2\pi) - n\log\theta - \sum_{i=1}^{n}\log X_i - \frac{1}{2\theta^2}\sum_{i=1}^{n}(\log X_i - \theta)^2 \\ &= -\frac{n}{2}\log(2\pi) - n\log\theta - \sum_{i=1}^{n}\log X_i - \frac{1}{2\theta^2}\sum_{i=1}^{n}\left[(\log X_i)^2 - 2\theta\log X_i + \theta^2\right] \\ &= -\frac{n}{2}\log(2\pi) - n\log\theta - \sum_{i=1}^{n}\log X_i - \frac{1}{2\theta^2}\left\{\left[\sum_{i=1}^{n}(\log X_i)^2\right] - 2\theta\left(\sum_{i=1}^{n}\log X_i\right) + n\theta^2\right\} \\ &= -\frac{n}{2}\log(2\pi) - n\log\theta - \sum_{i=1}^{n}\log X_i - \frac{1}{2\theta^2}\left[\sum_{i=1}^{n}(\log X_i)^2\right] + \frac{1}{\theta}\left(\sum_{i=1}^{n}\log X_i\right) - \frac{n}{2}. \end{aligned}$$

The derivative with respect to θ is

$$\frac{d\log L(\theta)}{d\theta} = -\frac{n}{\theta} + \frac{1}{\theta^3}\left[\sum_{i=1}^{n}(\log X_i)^2\right] - \frac{1}{\theta^2}\left(\sum_{i=1}^{n}\log X_i\right).$$

Setting this to zero, we obtain the quadratic equation

$$n\theta^2 + \left(\sum_{i=1}^{n}\log X_i\right)\theta - \left[\sum_{i=1}^{n}(\log X_i)^2\right] = 0. \tag{1}$$

Its roots are

$$\widehat{\theta} = \frac{-\left(\sum_{i=1}^{n}\log X_i\right) \pm \sqrt{\left(\sum_{i=1}^{n}\log X_i\right)^2 + 4n\sum_{i=1}^{n}(\log X_i)^2}}{2n}.$$

Since θ must be positive, the valid root is

$$\widehat{\theta} = \frac{-\left(\sum_{i=1}^{n} \log X_i\right) + \sqrt{\left(\sum_{i=1}^{n} \log X_i\right)^2 + 4n \sum_{i=1}^{n} \left(\log X_i\right)^2}}{2n}.$$

The second derivative of the log likelihood is

$$\begin{aligned} \frac{d^2 \log L(\theta)}{d\theta^2} &= \frac{n}{\theta^2} - \frac{3}{\theta^4} \left[\sum_{i=1}^{n} (\log X_i)^2\right] + \frac{2}{\theta^3} \left(\sum_{i=1}^{n} \log X_i\right) \\ &= \frac{1}{\theta^4} \left[n\theta^2 - 3 \left[\sum_{i=1}^{n} (\log X_i)^2\right] + 2\theta \left(\sum_{i=1}^{n} \log X_i\right)\right]. \end{aligned} \tag{2}$$

From (1),

$$\left(\sum_{i=1}^{n} \log X_i\right) \theta = -n\theta^2 + \left[\sum_{i=1}^{n} (\log X_i)^2\right] = 0. \tag{3}$$

By substituting (3) into (2), we see

$$\frac{d^2 \log L(\theta)}{d\theta^2} = -\frac{1}{\theta^4} \left\{n\theta^2 + \left[\sum_{i=1}^{n} (\log X_i)^2\right]\right\} < 0. \tag{4}$$

Hence,

$$\widehat{\theta} = \frac{-\left(\sum_{i=1}^{n} \log X_i\right) + \sqrt{\left(\sum_{i=1}^{n} \log X_i\right)^2 + 4n \sum_{i=1}^{n} \left(\log X_i\right)^2}}{2n}.$$

is a maximum likelihood estimator of θ.