MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 7

Suppose X_i are distributed as Poisson$(i\lambda)$ for $i = 1, 2, \ldots, n$. Suppose too $X_1, \ldots, X_n$ are independently distributed. The likelihood function of λ is

$$\begin{aligned} L(\lambda) &= \prod_{i=1}^{n} \frac{(i\lambda)^{X_i} \exp(-i\lambda)}{X_i!} \\ &= \left(\prod_{i=1}^{n} i^{X_i}\right) \lambda^{\sum_{i=1}^{n} X_i} \exp\left(-\lambda \sum_{i=1}^{n} i\right) \left(\prod_{i=1}^{n} X_i!\right)^{-1} \\ &= \left(\prod_{i=1}^{n} i^{X_i}\right) \lambda^{\sum_{i=1}^{n} X_i} \exp\left(-\frac{n(n+1)}{2}\lambda\right) \left(\prod_{i=1}^{n} X_i!\right)^{-1}. \end{aligned}$$

Its log is

$$\log L(\lambda) = \sum_{i=1}^{n} X_i \log i + \left(\sum_{i=1}^{n} X_i\right) \log \lambda - \frac{n(n+1)}{2}\lambda - \sum_{i=1}^{n} \log X_i!.$$

The derivative with respect to λ is

$$\frac{d \log L(\lambda)}{d\lambda} = \left(\sum_{i=1}^{n} X_i\right) \frac{1}{\lambda} - \frac{n(n+1)}{2}.$$

Setting this to zero and solving for λ, we obtain

$$\widehat{\lambda} = \frac{2}{n(n+1)} \left(\sum_{i=1}^{n} X_i\right).$$

This is a maximum likelihhood estimator since

$$\frac{d^2 \log L(\lambda)}{d\lambda^2} = -\left(\sum_{i=1}^{n} X_i\right) \frac{1}{\lambda^2} < 0.$$