MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 6

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the cumulative distribution function

$$F(x) = 1 - \left(\frac{K}{x}\right)^a$$

for $x > K > 0$ and $a > 0$. Let $Z = \min(X_1, X_2, \ldots, X_n)$. The cdf of Z is

$$\begin{aligned}
F_Z(z) &= \Pr(Z \le z) \\
&= 1 - \Pr(Z > z) \\
&= 1 - \Pr\left[\min(X_1, X_2, \ldots, X_n) > z\right] \\
&= 1 - \Pr\left(X_1 > z, X_2 > z, \ldots, X_n > z\right) \\
&= 1 - \Pr(X_1 > z) \Pr(X_2 > z) \cdots \Pr(X_n > z) \\
&= 1 - \left[1 - \Pr(X_1 \le z)\right]\left[1 - \Pr(X_2 \le z)\right] \cdots \left[1 - \Pr(X_n \le z)\right] \\
&= 1 - [1 - F(z)][1 - F(z)] \cdots [1 - F(z)] \\
&= 1 - [1 - F(z)]^n \\
&= 1 = \left(\frac{K}{z}\right)^{na}.
\end{aligned}$$

The corresponding density function is

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{naK^{na}}{z^{na+1}}$$

for $z > K$. Hence, the expected value of $\widehat{K}$ is

$$\begin{aligned}
E\left(\widehat{K}\right) &= E(Z) \\
&= \int_K^\infty z \frac{naK^{na}}{z^{na+1}} dz \\
&= naK^{na} \int_K^\infty z^{-na} dz \\
&= naK^{na} \left[\frac{z^{1-na}}{1-na}\right]_K^\infty \\
&= naK^{na} \left[0 - \frac{K^{1-na}}{1-na}\right] \\
&= \frac{naK}{na-1}.
\end{aligned}$$

The expected value of the square of $\widehat{K}$ is

$$\begin{aligned}
E\left(\widehat{K}^2\right) &= E\left(Z^2\right) \\
&= \int_K^\infty z^2 \frac{naK^{na}}{z^{na+1}} dz \\
&= naK^{na} \int_K^\infty z^{1-na} dz \\
&= naK^{na} \left[\frac{z^{2-na}}{2-na}\right]_K^\infty \\
&= naK^{na} \left[0 - \frac{K^{2-na}}{2-na}\right] \\
&= \frac{naK^2}{na-2}.
\end{aligned}$$

Hence, the variance of $\widehat{K}$ is

$$\begin{aligned}
Var\left(\widehat{K}\right) &= E\left(\widehat{K}^2\right) - \left[E\left(\widehat{K}\right)\right]^2 \\
&= \frac{naK^2}{na-2} - \frac{n^2a^2K^2}{(na-1)^2} \\
&= naK^2 \left[\frac{1}{na-2} - \frac{na}{(na-1)^2}\right] \\
&= \frac{naK^2}{(na-2)(na-1)^2}.
\end{aligned}$$

Hence, the MSE of $\widehat{K}$ is

$$MSE\left(\widehat{K}\right) = \frac{naK^2}{(na-2)(na-1)^2} + \left(\frac{K}{na-1}\right)^2,$$

which approaches 0 as $n \to \infty$. Hence, $\widehat{K}$ is consistent.