MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 5

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution specified by the cumulative distribution function

$$F(x) = 1 - \left(\frac{K}{x}\right)^a$$

for $x > K > 0$ and $a > 0$. Let $Z = \min\left(X_1, X_2, \ldots, X_n\right)$. The cdf of Z is

$$\begin{aligned}
F_Z(z) &= \Pr(Z \le z) \\
&= 1 - \Pr(Z > z) \\
&= 1 - \Pr\left[\min\left(X_1, X_2, \ldots, X_n\right) > z\right] \\
&= 1 - \Pr\left(X_1 > z, X_2 > z, \ldots, X_n > z\right) \\
&= 1 - \Pr\left(X_1 > z\right)\Pr\left(X_2 > z\right)\cdots\Pr\left(X_n > z\right) \\
&= 1 - \left[1 - \Pr\left(X_1 \le z\right)\right]\left[1 - \Pr\left(X_2 \le z\right)\right]\cdots\left[1 - \Pr\left(X_n \le z\right)\right] \\
&= 1 - [1 - F(z)][1 - F(z)]\cdots[1 - F(z)] \\
&= 1 - [1 - F(z)]^n \\
&= 1 - \left(\frac{K}{z}\right)^{na}.
\end{aligned}$$

The corresponding density function is

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{naK^{na}}{z^{na+1}}$$

for $z > K$. Hence, the bias of $\widehat{K}$ is

$$\begin{aligned}
Bias\left(\widehat{K}\right) &= E\left(\widehat{K}\right) - K \\
&= E(Z) - K \\
&= \int_K^\infty z\frac{naK^{na}}{z^{na+1}}dz - K \\
&= naK^{na}\int_K^\infty z^{-na}dz - K \\
&= naK^{na}\left[\frac{z^{1-na}}{1-na}\right]_K^\infty - K \\
&= naK^{na}\left[0 - \frac{K^{1-na}}{1-na}\right] - K \\
&= \frac{naK}{na-1} - K \\
&= \frac{K}{na-1}.
\end{aligned}$$

Hence, $\widehat{K}$ is biased.