MATH10282: INTRODUCTION TO STATISTICS
SEMESTER 2
SOLUTIONS TO QUIZ PROBLEM 10

Suppose $X_1, X_2, \ldots, X_n$ is an independent random sample from Exp (θ). Consider the test for $H_0 : \theta = \theta_1$ versus $H_1 : \theta = \theta_2$ with

$$\sum_{i=1}^{n} X_i < c$$

as the rule for rejecting H_0.

Let $Z = \sum_{i=1}^{n} X_i$. Suppose $n = 1$. Then the type II error probability is

$$\begin{aligned}
\Pr(\text{Type II Error}) &= \Pr(X_1 \geq c \mid \theta_2) \\
&= 1 - \Pr(X_1 < c \mid \theta_2) \\
&= 1 - \Pr(\text{Exp}(\theta_2) < c \mid \theta_2) \\
&= 1 - [1 - \exp(-\theta_2 c)] \\
&= \exp(-\theta_2 c),
\end{aligned}$$

so the statement holds when $n = 1$. Suppose now $n = 2$. Then the type II error probability is

$$\begin{aligned}
\Pr(\text{Type II Error}) &= \Pr(X_1 + X_2 \geq c \mid \theta_2) \\
&= 1 - \Pr(X_1 + X_2 < c \mid \theta_2) \\
&= 1 - \Pr(\text{Exp}(\theta_2) + \text{Exp}(\theta_2) < c \mid \theta_2) \\
&= 1 - \int_0^c \{1 - \exp[-\theta_2 (c - x)]\} \theta_2 \exp(-\theta_2 x)\, dx \\
&= 1 - \int_0^c \theta_2 \exp(-\theta_2 x) dx - \int_0^c \theta_2 \exp(-\theta_2 c)\, dx \\
&= 1 - [1 - \exp(-\theta_2 c) - \theta_2 c \exp(-\theta_2 c)] \\
&= (1 + \theta_2 c) \exp(-\theta_2 c),
\end{aligned}$$

so the statement holds when $n = 2$. Next, assume that the statement is true for $n = k - 1$, that is

$$\Pr(\text{Type II Error}) = \left[1 + \theta_2 c + \cdots + \frac{\theta_2^{k-2} c^{k-2}}{(k-2)!}\right] \exp(-\theta_2 c).$$

The statement also holds for $n = k$ since

$$\begin{aligned}
& \Pr(\text{Type II Error}) \\
= \; & \Pr(X_1 + \cdots + X_n \geq c \mid \theta_2)
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \Pr\left(X_1 + \cdots + X_n < c \mid \theta_2\right) \\
&= 1 - \int_0^c \left\{ 1 - \left[1 + \theta_2 (c - x) + \cdots + \frac{\theta_2^{k-2} (c - x)^{k-2}}{(k-2)!} \right] \exp\left[-\theta_2 (c - x)\right] \right\} \theta_2 \exp\left(-\theta_2 x\right) dx \\
&= 1 - \int_0^c \theta_2 \exp\left(-\theta_2 x\right) dx + \theta_2 \exp\left(-\theta_2 c\right) \int_0^c \left[1 + \theta_2 (c - x) + \cdots + \frac{\theta_2^{k-2} (c - x)^{k-2}}{(k-2)!} \right] dx \\
&= 1 - \int_0^c \theta_2 \exp\left(-\theta_2 x\right) dx + \theta_2 \exp\left(-\theta_2 c\right) \left[x - \theta_2 \frac{(c - x)^2}{2} - \cdots - \frac{\theta_2^{k-2} (c - x)^{k-1}}{(k-1)!} \right]_0^c \\
&= \exp\left(-\theta_2 c\right) + \theta_2 \exp\left(-\theta_2 c\right) \left[c + \theta_2 \frac{c^2}{2} + \cdots + \frac{\theta_2^{k-2} c^{k-1}}{(k-1)!} \right] \\
&= \exp\left(-\theta_2 c\right) + \exp\left(-\theta_2 c\right) \left[\theta_2 c + \frac{\theta_2^2 c^2}{2} + \cdots + \frac{\theta_2^{k-1} c^{k-1}}{(k-1)!} \right] \\
&= \left[1 + \theta_2 c + \frac{\theta_2^2 c^2}{2} + \cdots + \frac{\theta_2^{k-1} c^{k-1}}{(k-1)!} \right] \exp\left(-\theta_2 c\right).
\end{aligned}
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Hence, the result follows.