Extrmal Types Theorem (ETT)

Suppose $X_1, \ldots, X_n$ are IID with CDF F. Let $M_n = \max(X_1, \ldots, X_n)$. If there exists $a_n > 0$, $b_n \in \mathbb{R}$ & a non-degenerate CDF G such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \xrightarrow[n \to \infty]{} G(x)$$

then G must be of the same type as

$$\text{I}: \quad \Lambda(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$$

$$\text{II}: \quad \Phi_\alpha(x) = e^{-x^{-\alpha}}, \quad x > 0$$

$$\text{III}: \quad \Psi_\alpha(x) = e^{-(-x)^{\alpha}}, \quad x < 0$$

Given an F, which type would it belong to?

$$\mathrm{I}: \exists\, \gamma(t) \text{ s.t. } \lim_{t \uparrow \omega(F)} \frac{1-F(t+x\gamma(t))}{1-F(t)} = e^{-x}$$

$$\mathrm{II}: \omega(F) = \infty \ \& \lim_{t \uparrow \infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha}$$

$$\mathrm{III}: \omega(F) < \infty \ \& \lim_{t \downarrow 0} \frac{1-F(\omega(F)-tx)}{1-F(\omega(F)-t)} = x^{\alpha}$$

How to choose a_n & b_n?

I : $a_n = \gamma\left(F^{\leftarrow}\left(1-\frac{1}{n}\right)\right)$ & $b_n = F^{\leftarrow}\left(1-\frac{1}{n}\right)$

II : $a_n = F^{\leftarrow}\left(1-\frac{1}{n}\right)$ & $b_n = 0$

III : $a_n = w(F) - F^{\leftarrow}\left(1-\frac{1}{n}\right)$ & $b_n = w(F)$

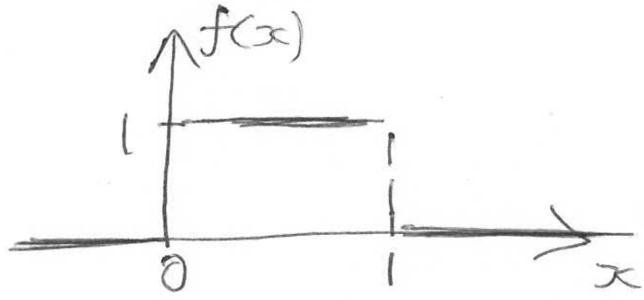
Defn $\quad w(F) = \sup\{x : F(x) < 1\}$

"upper end point" of F

eg

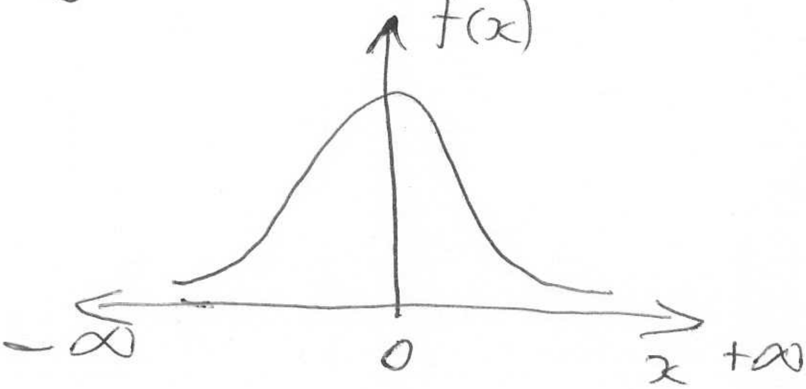
- if F is the CDF of Uni $[0,1]$

then $w(F) = 1$



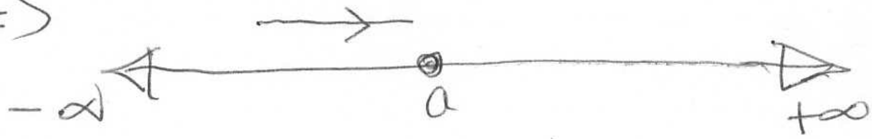
- if F is the CDF of $N(0,1)$

then $w(F) = +\infty$

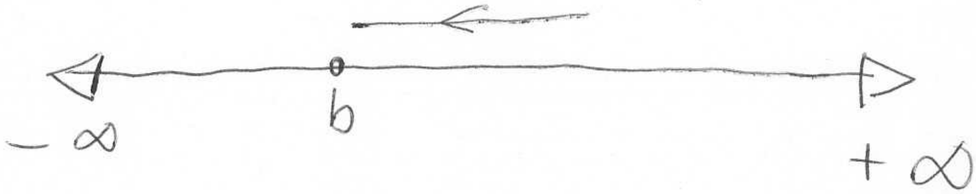


Defn

$t \uparrow a \iff$



$t \downarrow b \iff$



Defn

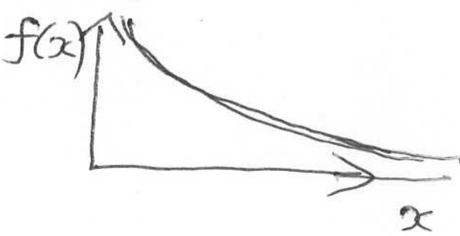
Let $F^{\leftarrow}$ denote the inverse function of F

(usually F^{-1} is used for the inverse function. But this notation can be confusing : F^{-1} could also mean $\frac{1}{F}$)

Ex 1 $\quad F(x) = 1 - e^{-x}, \quad x > 0$

(Exponential distribution)

$$\omega(F) = \infty$$



$$\text{I}: \lim_{t \uparrow \omega(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{1} - (\cancel{1} - e^{-t - x\gamma(t)})}{\cancel{1} - (\cancel{1} - e^{-t})}$$

$$= \lim_{t \uparrow \infty} \frac{e^{-t - x\gamma(t)}}{e^{-t}}$$

$$= e^{-x\gamma(t)}$$

$$= e^{-x} \quad \text{if } \gamma(t) = 1$$

So, condition I is satisfied

$$F(x) = 1 - e^{-x}$$

$$F^{\leftarrow}(x) = -\log(1-x)$$

$$F^{\leftarrow}\left(1-\frac{1}{n}\right) = -\log\left(1-\left(1-\frac{1}{n}\right)\right)$$

$$= -\log\left(\frac{1}{n}\right)$$

$$= \log n$$

$$a_n = \gamma\left(F^{\leftarrow}\left(1-\frac{1}{n}\right)\right) = \underline{\underline{1}}$$

$$b_n = F^{\leftarrow}\left(1-\frac{1}{n}\right) = +\log n$$

Hence,

$$P\left(\frac{M_n - \log n}{1} < x\right) \to e^{-e^{-x}}$$

as $n \to \infty$

by the ETT.

Ex 2

$$F(x) = x \ , \quad 0 < x < 1$$

(Uniform [0, 1] distribution)

$$\omega(F) = 1$$

I : $\lim\limits_{t \uparrow 1} \dfrac{1 - F(t + x\gamma(t))}{1 - F(t)}$

$$= \lim_{t \uparrow 1} \frac{1 - (t + x\gamma(t))}{1 - t}$$

$$= \lim_{t \uparrow 1} \left[1 - \frac{x\gamma(t)}{1-t} \right]$$

$$= 1 - \frac{x\gamma(1)}{0} \neq e^{-x}$$

$\Rightarrow$ I is not satisfied

$\underline{\overline{II}}$: $w(F) = 1 \neq \infty$

$\Rightarrow$ Condition $\underline{\overline{II}}$ not satisfied

$\underline{\overline{\text{III}}}$:

$$\lim_{t \downarrow 0} \frac{1 - F(1 - tx)}{1 - F(1-t)}$$

$$= \lim_{t \downarrow 0} \frac{\cancel{1} - (\cancel{1} - tx)}{\cancel{1} - (\cancel{1} - t)}$$

$$= \lim_{t \downarrow 0} \frac{tx}{t} = x$$

$\Rightarrow$ Condition $\underline{\overline{\text{III}}}$ is satisfied

$$F(x) = x \Rightarrow F^{\leftarrow}(x) = x$$

$$a_n = \omega(F) - F^{\leftarrow}\left(1 - \frac{1}{n}\right) = 1 - \left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

$$b_n = \omega(F) = 1$$

By the ETT,

$$P\left(\frac{M_n - 1}{1/n} < x \right) \longrightarrow e^{-(-x)^1}$$

as $n \rightarrow \infty$.

Ex 3

$$F(x) = 1 - \left(\frac{K}{x}\right)^a, \quad x \geq K$$

(Pareto distribution)

$$\omega(F) = \infty$$

I:

$$\lim_{t \uparrow \infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{1} - \left[\cancel{1} - \left(\frac{K}{t + x\gamma(t)}\right)^a\right]}{\cancel{1} - \left[\cancel{1} - \left(\frac{K}{t}\right)^a\right]}$$

$$= \lim_{t \uparrow \infty} \left(\frac{t}{t + x\gamma(t)}\right)^a$$

$$= \lim_{t \uparrow \infty} \left(\frac{1}{1 + x\gamma(t)/t}\right)^a \neq e^{-x}$$

$\Rightarrow$ Condition I not satisfied

$\underline{\overline{II}}$:

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{1} - \left[\cancel{1} - \left(\frac{k}{tx}\right)^a\right]}{\cancel{1} - \left[\cancel{1} - \left(\frac{k}{t}\right)^a\right]}$$

$$= \lim_{t \uparrow \infty} \left(\frac{t}{tx}\right)^a = x^{-a}$$

$\Rightarrow$ Condition $\underline{II}$ is satisfied

$$F(x) = 1 - \left(\frac{k}{x}\right)^a \Rightarrow F^{\leftarrow}(x) = k(1-x)^{-\frac{1}{a}}$$

$$a_n = F^{\leftarrow}\left(1 - \frac{1}{n}\right) = k\left(1 - \left(1 - \frac{1}{n}\right)\right)^{-\frac{1}{a}} = k n^{\frac{1}{a}}$$

$b_n = 0$

By the ETT,

$$P\left(\frac{M_n - 0}{k n^{1/a}} < x\right) \longrightarrow e^{-x^{-a}}$$

as $n \to \infty$

The Extremal types theorem

Lemma 1. *If G is max-stable, then there exist real-valued functions $a(s) > 0$ and $b(s)$, defined for $s > 0$, such that*

$$G^n(a(s)x + b(s)) = G(x).$$

Proof. Since G is max-stable, there exist $a_n > 0$ and b_n such that

$$G^s(a_n x + b_n) = G(x) \xrightarrow{d} G(x).$$

Thus $G^{\lfloor ns \rfloor}(a_{\lfloor ns \rfloor}x + b_{\lfloor ns \rfloor}) = G(x)$, and we deduce that

$$G^n(a_{\lfloor ns \rfloor}x + b_{\lfloor ns \rfloor}) = \exp\Big\{\frac{n}{\lfloor ns \rfloor}\lfloor ns \rfloor \log G(a_{\lfloor ns \rfloor}x + b_{\lfloor ns \rfloor})\Big\} \xrightarrow{d} G^{1/s}(x).$$

Since $G^{1/s}$ is non-degenerate, the lemma from lectures gives that there exist $a(s) > 0$ and $b(s)$ such that $G(a(s)x + b(s)) = G^{1/s}(x)$, so $G^s(a(s)x + b(s)) = G(x)$. □

Theorem 2 (Extremal types theorem). *Let (X_n) be independent with distribution function F and let $X_{(n)} = \max_{1 \le i \le n} X_{(i)}$. If there exist constants $a_n > 0$ and b_n and a non-degenerate distribution function G such that*

$$\mathbb{P}\Big(\frac{X_{(n)} - b_n}{a_n} \le x\Big) \xrightarrow{d} G(x),$$

then G must be of the same type as one of the three extreme value *classes below:*

Type I *(Fréchet): $G_{1,\alpha}(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \exp(-x^{-\alpha}) & \text{if } x > 0 \end{cases}$ for some $\alpha > 0$*

Type II *(Negative Weibull): $G_{2,\alpha}(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$ for some $\alpha > 0$*

Type III *(Gumbel): $G_3(x) = \exp(-e^{-x})$ for $x \in \mathbb{R}$.*

Conversely, any distribution function of the same type as one of these extreme value classes can appear as such a limit.

Proof. It suffices to show that the class of max-stable distribution functions coincides with the set of distribution functions of the same type as the three given extreme value classes. To check that the given distribution functions are max-stable, it suffices to observe that if $a_n = n^{1/\alpha}$, $b_n = 0$, then

$$G^n_{1,\alpha}(a_n x + b_n) = \begin{cases} 0 & \text{if } x \le 0 \\ \exp\{-n(a_n x + b_n)^{-\alpha}\} & \text{if } x > 0 \end{cases} = G_{1,\alpha}(x).$$

Similarly, if $a_n = n^{-1/\alpha}$, $b_n = 0$, then

$$G^n_{2,\alpha}(a_n x + b_n) = \begin{cases} \exp\{-n(-a_n x - b_n)^{\alpha}\} & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases} = G_{2,\alpha}(x).$$

Finally, if $a_n = 1$, $b_n = \log n$, then

$$G_3(a_n x + b_n) = \exp\{-ne^{-(a_n x + b_n)}\} = \exp(-e^{-x}).$$

Conversely, suppose G is max-stable, so by Lemma 1 we can write $G^s(a(s)x+b(s)) = G(x)$. It follows that for $0 < G(x) < 1$,

$$-\log\{-\log G(a(s)x + b(s))\} - \log s = \log\{-\log G(x)\}.$$

The max-stability property with $n = 2$ gives that $G^2(ax + b) = G(x)$ for some $a > 0$ and $b \in \mathbb{R}$, which means G cannot have a jump at $x_- = \sup\{x : G(x) = 0\}$ or $x_+ = \inf\{x : G(x) = 1\}$ if these are finite. Thus the non-decreasing function $\psi(x) = -\log\{-\log G(x)\}$ is such that

$$\lim_{x \to x_-} \psi(x) = -\infty, \qquad \lim_{x \to x_+} \psi(x) = \infty.$$

Therefore ψ has an inverse function $U(y) = \inf\{x \in \mathbb{R} : \psi(x) \ge y\}$, defined for all $y \in \mathbb{R}$, and since $\psi(a(s)x + b(s)) - \log s = \psi(x)$, it follows that

$$\begin{aligned} U(y) &= \inf\{x : \psi(a(s)x + b(s)) - \log s \ge y\} \\ &= \frac{1}{a(s)}\{\inf\{x' : \psi(x') \ge y + \log s\} - b(s)\} \\ &= \frac{U(y + \log s) - b(s)}{a(s)}. \end{aligned}$$

Subtracting this equation for $y = 0$,

$$\frac{U(y + \log s) - U(\log s)}{a(s)} = U(y) - U(0),$$

and writing $z = \log s$, $\tilde{a}(z) = a(e^z)$ and $\tilde{U}(y) = U(y) - U(0)$,

$$\tilde{U}(y + z) - \tilde{U}(z) = \tilde{U}(y)\tilde{a}(z) \tag{1}$$

for all $y, z \in \mathbb{R}$. Interchanging y and z and subtracting,

$$\tilde{U}(y)\{1 - \tilde{a}(z)\} = \tilde{U}(z)\{1 - \tilde{a}(y)\}. \tag{2}$$

Two cases are possible:

i) $\tilde{a}(z_0) \neq 1$ for some $z_0 > 0$. Then $\tilde{a}(z) \neq 1$ for all $z > 0$, because otherwise there exists $z > 0$ such that $\tilde{U}(z) = 0$. But this would mean that $\tilde{U}(y+z) = \tilde{U}(y)$ for all y, by (1), so $U(y+z) = U(y)$ for all $y \in \mathbb{R}$, a contradiction. Fixing $z > 0$, writing $c = \tilde{U}(z)/\{1 - \tilde{a}(z)\}$ and noting from (2) that this is constant, we have from (1) that

$$c(1 - \tilde{a}(y+z)) - c(1 - \tilde{a}(z)) = c(1 - \tilde{a}(y))\tilde{a}(z),$$

so that

$$\tilde{a}(y+z) = \tilde{a}(y)\tilde{a}(z)$$

for all $y \in \mathbb{R}$. But $\tilde{a}$ is monotone, since $\tilde{U}(y) = c\{1 - \tilde{a}(y)\}$ from (2), and the only non-zero solutions that are monotone and not identically equal to 1 are $\tilde{a}(y) = e^{\rho y}$ for some $\rho \neq 0$ (check). But then

$$\psi^{-1}(y) = U(y) = \nu + c(1 - e^{\rho y})$$

where $\nu = U(0)$. Since ψ^{-1} is non-decreasing, we must have $c < 0$ if $\rho > 0$ and $c > 0$ if $\rho < 0$, so in fact ψ^{-1} is continuous and strictly increasing. Hence

$$x = \psi^{-1}(\psi(x)) = \nu + c(1 - e^{\rho\psi(x)}) = \nu + c[1 - \{-\log G(x)\}^{-\rho}],$$

so

$$G(x) = \exp\left\{-\left(1 - \frac{x - \nu}{c}\right)^{-1/\rho}\right\}$$

for $0 < G(x) < 1$. From the continuity of G at any finite endpoints, we see that G is of Type I, with $\alpha = 1/\rho$, if $\rho > 0$, and of Type II, with $\alpha = -1/\rho$, if $\rho < 0$.

ii) $\tilde{a}(z) = 1$ for all $z > 0$. But then, from (1),

$$\tilde{U}(y+z) = \tilde{U}(y) + \tilde{U}(z),$$

for which the only non-constant non-decreasing solutions are $\tilde{U}(y) = \rho y$ for some $\rho > 0$. Thus

$$\psi^{-1}(y) = U(y) = \nu + \rho y,$$

where $\nu = U(0)$, and since this is continuous and strictly increasing,

$$x = \psi^{-1}(\psi(x)) = \rho\psi(x) + \nu = -\rho\log\{-\log G(x)\} + \nu.$$

Hence $G(x) = \exp\{-e^{-(x-\nu)/\rho}\}$ for $0 < G(x) < 1$, and since G has no jump at any finite endpoint, G is of Type III. □

P11 Sheet I

(i) if G belongs to Gumbel domain

then F " " " "

(ii) if G belongs to Fréchet domain

then F " " " "

(iii) if G belongs to Weibull domain

then F " " " "

L'Hopital's Rule

$$\boxed{\lim \frac{f_1(x)}{f_2(x)} = \lim \frac{f_1'(x)}{f_2'(x)}}$$

(i) Suppose G belongs to Gumbel domain.

$$\exists\, \gamma(t) \text{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F(t+\gamma(t)x)}{1 - F(t)} = e^{-x}$$

L'Hopital's

$$\Rightarrow \lim_{t \uparrow w(F)} \frac{\cancel{-} f(t+\gamma(t)x)\,(1+\gamma'(t)x)}{\cancel{-}\, f(t)} = e^{-x}$$

$$\Rightarrow \lim_{t \uparrow w(F)} \frac{f(t+\gamma(t)x)}{f(t)}\,(1+\gamma'(t)x) = e^{-x}$$

$$\Rightarrow \cancel{\lim}\ \lim_{t \uparrow w(F)} \frac{K\, g(t+\gamma(t)x)\,\big(\underbrace{G(t+\gamma(t)x)}_{\to 1}\big)^{a-1} \cdot \{1 - G(t+\gamma(t)x)\}^{b-1} e^{-c\,\underbrace{G(t+\gamma(t)x)}_{\to 1}} \cdot (1+\gamma'(t)x)}{K\, g(t)\,\big(\underbrace{G(t)}_{\to 1}\big)^{a-1} \{1 - G(t)\}^{b-1} \cdot e^{-c\,\underbrace{G(t)}_{\to 1}}}$$

$$= e^{-x}$$

$$\Rightarrow \lim_{t \uparrow w(F)} \frac{\cancel{K}\, g(t+\gamma(t)x)\,\{1-G(t+\gamma(t)x)\}^{b-1}\,\cancel{e^{-c}} \cdot (1+\gamma'(t)x)}{\cancel{K}\, g(t)\,\{1-G(t)\}^{b-1}\,\cancel{e^{-c}}}$$

$$= e^{-x}$$

$$\Rightarrow \lim_{t \uparrow w(F)} \frac{g(t+\gamma(t)x)(1+\gamma'(t)x)}{g(t)} \cdot \left[\frac{1-G(t+\gamma(t)x)}{1-G(t)}\right]^{b-1} = e^{-x}$$

L'Hopital's

$$\Rightarrow \lim_{t \uparrow w(F)} \frac{1-G(t+\gamma(t)x)}{1-G(t)} \cdot \left[\frac{1-G(t+\gamma(t)x)}{1-G(t)}\right]^{b-1} = e^{-x}$$

$$\Rightarrow \lim_{t \uparrow w(F)} \left[\frac{1-G(t+\gamma(t)x)}{1-G(t)}\right]^{b} = e^{-x}$$

$$\Rightarrow \left(e^{-x}\right)^b = e^{-bx}$$

same type as e^{-x}

$\Rightarrow$ F belongs to Gumbel domain.

There are many examples where F does not satisfy

I, II & III

e.g.

$$F(x) = 1 - \frac{1}{\log x}, \quad x > e$$

(I) $w(F) = \infty$

$$\lim_{t\uparrow\infty} \frac{1-F(t+x\gamma(t))}{1-F(t)} = \lim \frac{\cancel{1} - \left(\cancel{1} - \frac{1}{\log(t+x\gamma(t))}\right)}{\cancel{1} - \left(\cancel{1} - \frac{1}{\log t}\right)}$$

$$= \lim_{t\uparrow\infty} \frac{\log t}{\log(t+x\gamma(t))}$$

$$= \lim_{t\uparrow\infty} \frac{1}{1 + \frac{\log\left(1+\frac{x\gamma(t)}{t}\right)}{\log t}}$$

$$\neq e^{-x}$$

$\Rightarrow$ Cond I not satisfied

II :

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \uparrow \infty} \frac{\cancel{1} - \left(\cancel{1} - \frac{1}{\log(tx)}\right)}{\cancel{1} - \left(\cancel{1} - \frac{1}{\log t}\right)}$$

$$= \lim_{t \uparrow \infty} \frac{\log t}{\log(tx)}$$

$$= \lim_{t \uparrow \infty} \frac{\log t}{\log t + \log x} = 1 \neq x^{\alpha}$$

$\Rightarrow$ Cond II not satisfied

$\overline{\underline{III}}$:

$w(F) = \infty$

not finite

$\Rightarrow$ Cond $\overline{\underline{III}}$ not satisfied

$\Rightarrow$ None of I-$\overline{\underline{III}}$ are satisfied,

$$\Rightarrow P\left(\frac{M_n - b_n}{a_n} < x\right) \not\rightarrow G(x)$$

for a non-degenerate $G(x)$.

Suppose F is a CDF of a continuous RV. Then ETT will not hold (that is, none of I – III will be satisfied) if

PDF

$$\lim_{x \uparrow w(F)} \frac{f(x)}{1 - F(x^-)} = 0$$

CDF

Suppose F is a CDF of a discrete RV. Then ETT will not hold (that is, none of contitions I–III will be satisfied) if

$$\lim_{k \uparrow w(F)} \frac{P(X=k)}{1-F(k-1)} \neq 0$$

Ex Geometric distn

$$P(X=k) = p(1-p)^{k-1}, \quad k \geq 1$$

$$F(k) = 1-(1-p)^k$$

$$\frac{P(X=k)}{1-F(k-1)}$$

$$= \frac{p(1-p)^{k-1}}{\cancel{1}-[\cancel{1}-(1-p)^{k-1}]}$$

$$= \frac{p(1-p)^{k-1}}{(1-p)^{k-1}}$$

$$= p \neq 0$$

$\Rightarrow$ None of (I)–(III) will be satisfied

Q1 Sheet 2

k	$P(X=k)$
0	$1-p$
1	p

$\leftarrow$ PMF

k	$F(k)$
0	$1-p$
1	1

$\leftarrow$ CDF

$$\frac{P(X=k)}{1-F(k-1)} = \begin{cases} \dfrac{1-p}{1-0} & k=0 \\ \dfrac{p}{1-(1-p)} & k=1 \end{cases}$$

$$= \begin{cases} 1-p & k=0 \\ 1 & k=1 \end{cases}$$

$$\neq 0$$

$\Rightarrow$ ETT will not hold

$\Rightarrow$ There are no a_n & b_n for which $\frac{M_n - b_n}{a_n}$ will have a non-degenerate limit

Q2, Sheet 2

$$\boxed{p(k) = \begin{cases} 1 & k = k_0 \\ 0 & k \neq k_0 \end{cases}} \leftarrow \text{PMF}$$

$$F(k) = \begin{cases} 0 & \text{if } k < k_0 \\ 1 & \text{if } k \geq k_0 \end{cases}$$

$$1 - F(k-1) = \begin{cases} 1 & \text{if } k-1 < k_0 \\ 0 & \text{if } k-1 \geq k_0 \end{cases}$$

~~R(k)~~

$$= \begin{cases} 1 & \text{if } k < 1 + k_0 \\ 0 & \text{if } k \geq 1 + k_0 \end{cases}$$

$$\frac{p(k)}{1-F(k-1)} = \begin{cases} \frac{0}{1} & k < k_0 \\ \frac{1}{1} & k = k_0 \\ \frac{0}{1} & k_0 < k < l + k_0 \\ \frac{0}{0} & k \geq l + k_0 \end{cases}$$

$$\neq 0$$

$\Rightarrow$ ETT will not hold

Q4, Sheet 2

$$p(k) = \frac{k^{-s}}{\zeta(s)}, \quad k \geq 1$$

$$1 - F(k-1) = \sum_{x=k}^{\infty} p(x)$$

$$= \sum_{x=k}^{\infty} \frac{x^{-s}}{\zeta(s)}$$

ANALYSIS → $\approx \int_k^{\infty} \frac{x^{-s}}{\zeta(s)}\, dx$

$$= \frac{1}{\zeta(s)} \left[\frac{x^{1-s}}{1-s} \right]_k^{\infty}$$

$$= \frac{1}{\zeta(s)} \left[0 - \frac{k^{1-s}}{1-s} \right] \text{ if } s>1$$

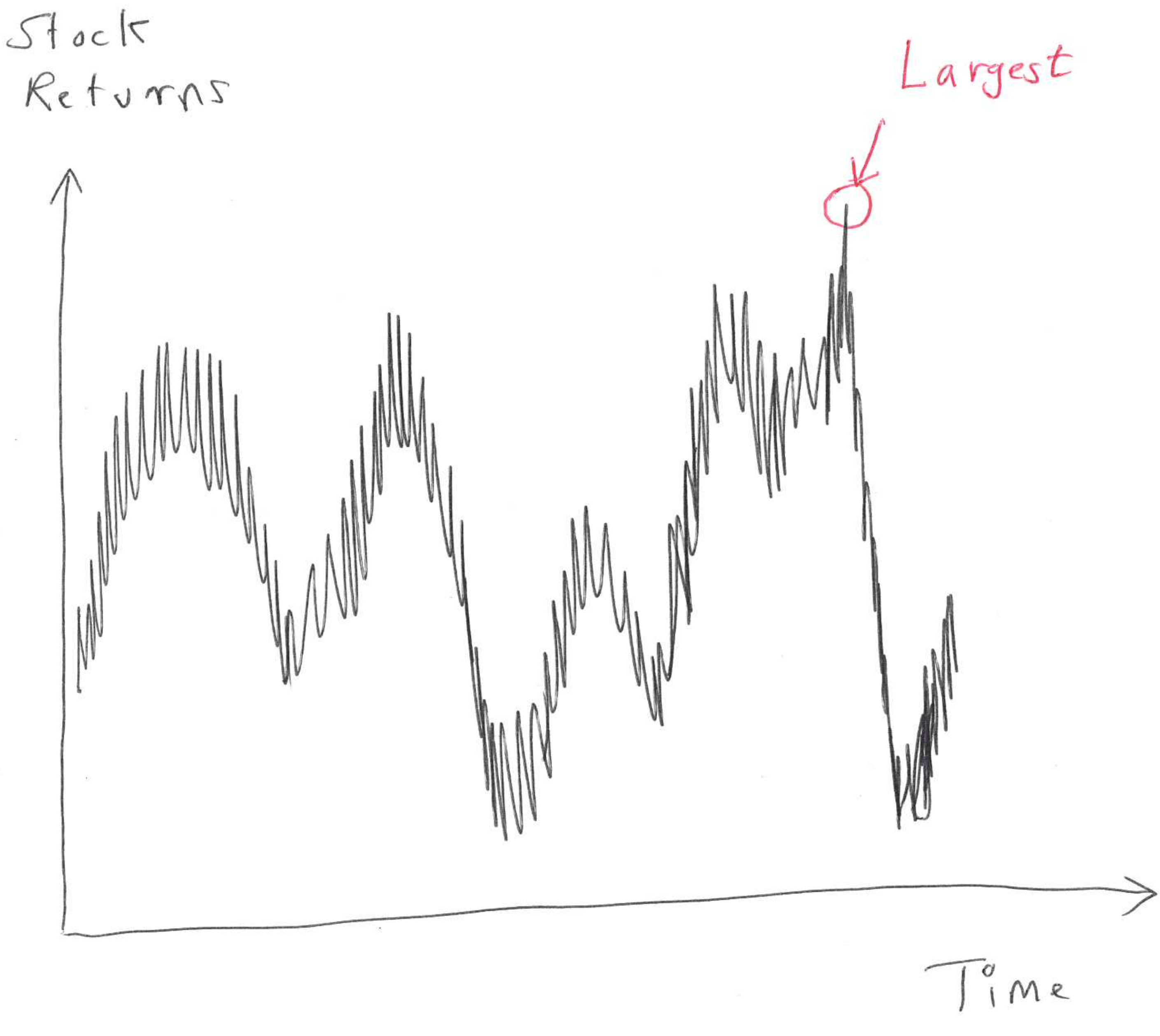
$$= \frac{1}{\zeta(s)} \frac{k^{1-s}}{s-1}$$

$$\lim \frac{p(k)}{1-F(k-1)} = \lim \frac{\dfrac{k^{-s}}{\cancel{\zeta(s)}}}{\dfrac{1}{\cancel{\zeta(s)}} \dfrac{k^{1-s}}{s-1}}$$

$$= \lim_{k \uparrow \infty} \frac{s-1}{k}$$

$$= 0$$

$\Rightarrow$ ETT will hold.


Stock Returns
Largest
Time

What is the use of ETT?

F is <u>known</u>

In practice,
F is <u>unknown</u>

What we need is a distribution that contains Gumbel, Frechet & Weibull as particular cases.

$\underline{\underline{\xi = 0}}$

$$G(x) \longrightarrow \boxed{e^{-e^{-\frac{x-\mu}{\sigma}}}}$$

$$\left[\left(1+\frac{x}{n}\right)^n \longrightarrow e^x \quad \text{as } n \to \infty \right]$$

same $\underline{\underline{\text{type}}}$ as Gumbel

$\underline{\underline{\xi > 0}}$

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$= e^{-\left(\frac{\xi}{\sigma}\right)^{-\frac{1}{\xi}}\left(\frac{\sigma}{\xi}+x-\mu\right)^{-\frac{1}{\xi}}}$$

Same $\underline{\text{type}}$ as Fréchet

$\underline{\xi < 0}$

$$G(x) = e^{-\left(-\frac{\xi}{\sigma}\right)^{-\frac{1}{\xi}}\left(-x+\left(1-\frac{\xi\mu}{\sigma}\right)\left(\frac{-\sigma}{\xi}\right)\right)^{-\frac{1}{\xi}}}$$

same <u>type</u> as

<u>Weibull</u>

$\Rightarrow$ GEV contains Gumbel, Fréchet & Weibull as particular cases.

The distribution satisfying this the <u>GEV</u> distn with CDF

$$G(x) = e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}},$$

$-\infty < \mu < \infty$ "location"

$\sigma > 0$ "scale"

$-\infty < \xi < \infty$ "shape"

$$1+\xi\,\frac{x-\mu}{\sigma} > 0$$

PDF

$$g(x) = \frac{1}{\sigma}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

Quantile

$$G(x_T) = 1 - \frac{1}{T}$$

T-year return level

no of years

" the level that is attained on average once in ~~T~~ years "

$$G(x_T) = 1 - \frac{1}{T}$$

$$\Rightarrow e^{-\left(1+\xi \frac{x_T-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} = 1 - \frac{1}{T}$$

$$\Rightarrow \left(1+\xi \frac{x_T-\mu}{\sigma}\right)^{-\frac{1}{\xi}} = -\log\left(1-\frac{1}{T}\right)$$

$$\Rightarrow 1+\xi \cdot \frac{x_T-\mu}{\sigma} = \left[-\log\left(1-\frac{1}{T}\right)\right]^{-\xi}$$

$$\Rightarrow \boxed{x_T = \mu + \frac{\sigma}{\xi}\left\{\left[-\log\left(1-\frac{1}{T}\right)\right]^{-\xi} - 1\right\}}$$

T - year return level.

Estimation

MLE

Suppose $X_1, \ldots, X_n$ is a random sample from GEV.

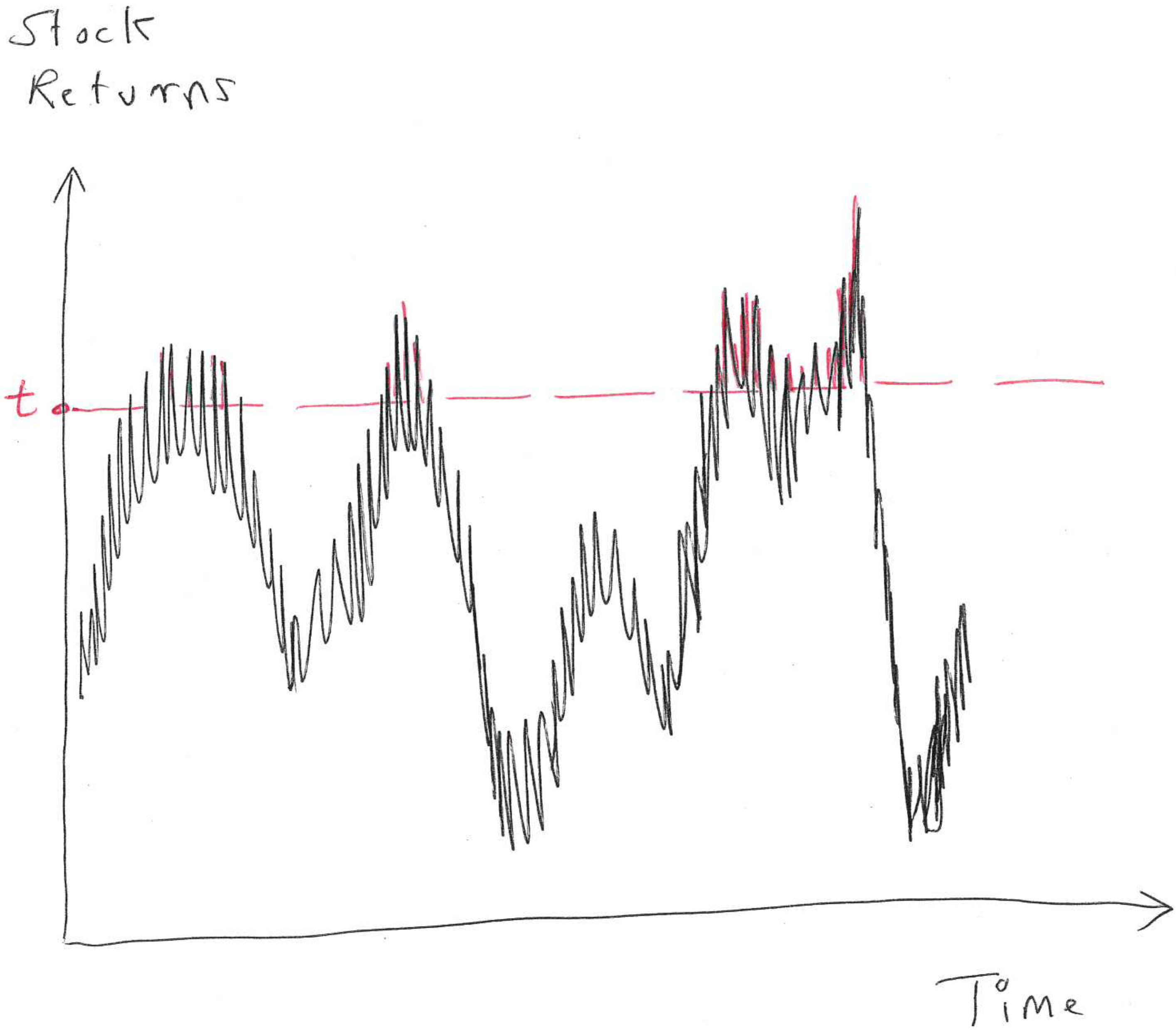
$$L(\mu, \sigma, \xi) = \prod_{i=1}^{n} \left[\frac{1}{\sigma} e^{-\left(1+\xi \frac{X_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} \cdot \left(1+\xi \frac{X_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1} \right]$$

$$= \frac{1}{\sigma^n} e^{-\sum_{i=1}^{n}\left(1+\xi \frac{X_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} \cdot \left[\prod_{i=1}^{n}\left(1+\xi \frac{X_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1}$$

$$\log L = -n\log\sigma - \sum_{i=1}^{n}\left(1+\xi \frac{X_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}} - \left(\frac{1}{\xi}+1\right)\sum_{i=1}^{n}\log\left(1+\xi \frac{X_i-\mu}{\sigma}\right) \quad (*)$$

MLEs say $(\hat{\mu}, \hat{\sigma}, \hat{\xi})$ are the values of (μ, σ, ξ) maximising $(*)$.

In R, this can be done using <u>optim</u> or <u>nlm</u>



Stock Returns
t₀
Time

How can we model the exceedances?

Thanks to Pickands (1975),

$$F(x) = 1 - w\left[1 + \xi \frac{x-t}{\sigma}\right]^{-\frac{1}{\xi}}$$

$w = P(X > t)$

$-\infty < \xi < \infty$ "shape"

$\sigma > 0$ "scale"

t threshold

$$1 + \xi \frac{X-t}{\sigma} > 0$$

" Generalized Pareto Distn "
(GP)

PDF

$$f(x) = \frac{w}{\sigma}\left[1 + \xi\,\frac{x-t}{\sigma}\right]^{-\frac{1}{\xi}-1}$$

Quantile

$$F(x_T) = 1 - \frac{1}{mT}$$

ave no of exceedances per year

no of years

T - yr return level

"level exceeded on average once in every T years"

$$F(x_T) = 1 - \frac{1}{mT}$$

$$\Rightarrow \cancel{1} - w\left[1 + \xi \frac{x_T - t}{\sigma}\right]^{-\frac{1}{\xi}} = \cancel{1} - \frac{1}{mT}$$

$$\Rightarrow \left[1 + \xi \frac{x_T - t}{\sigma}\right]^{-\frac{1}{\xi}} = \frac{1}{w\,mT}$$

$$\Rightarrow 1 + \xi \frac{x_T - t}{\sigma} = (w\,mT)^{\xi}$$

$$\Rightarrow \boxed{x_T = t + \frac{\sigma}{\xi}\left[(w\,mT)^{\xi} - 1\right]}$$

T-yr return level.

Estimation

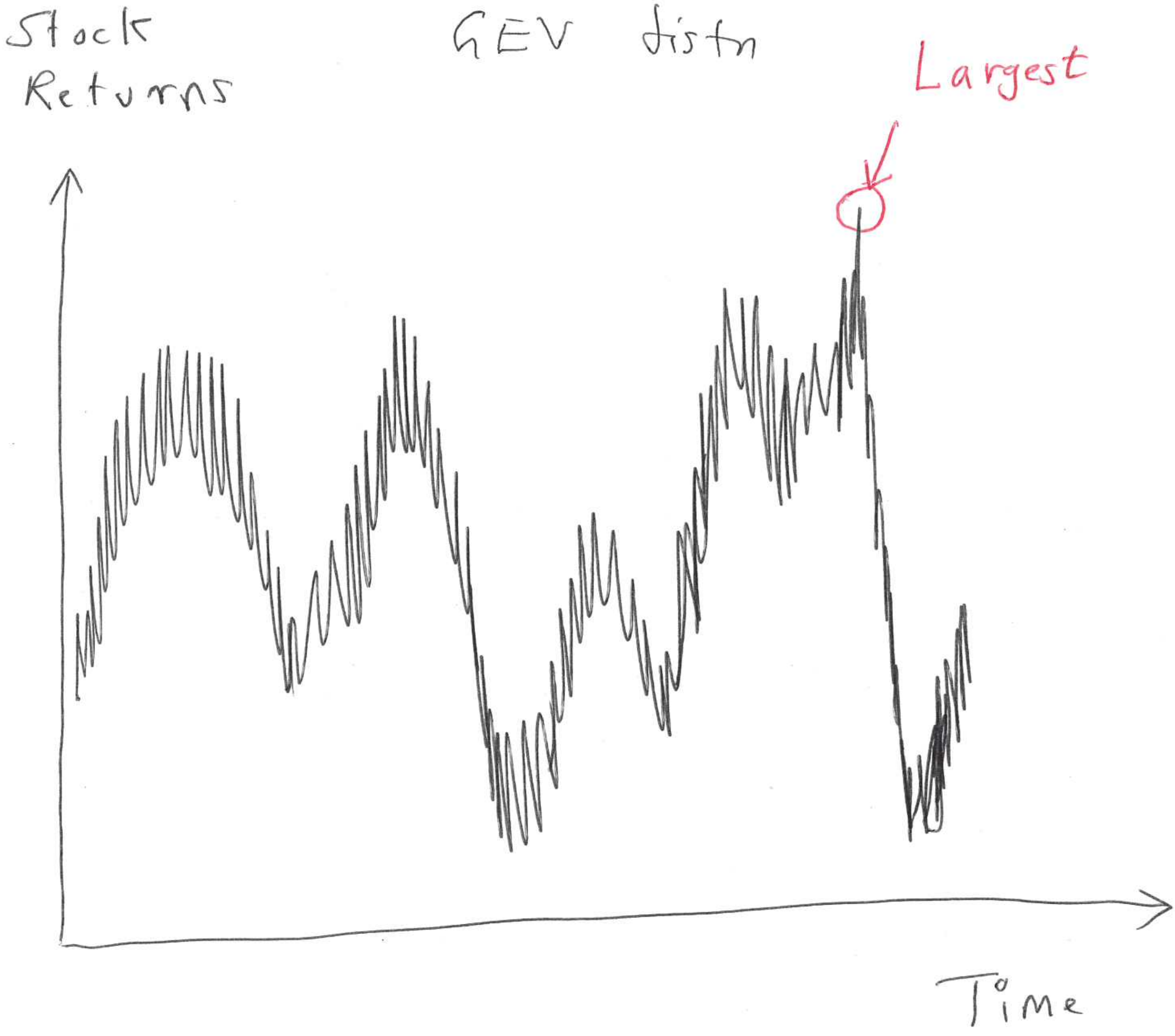
Suppose $X_1, \dots, X_n$ is a random sample from GP distn.

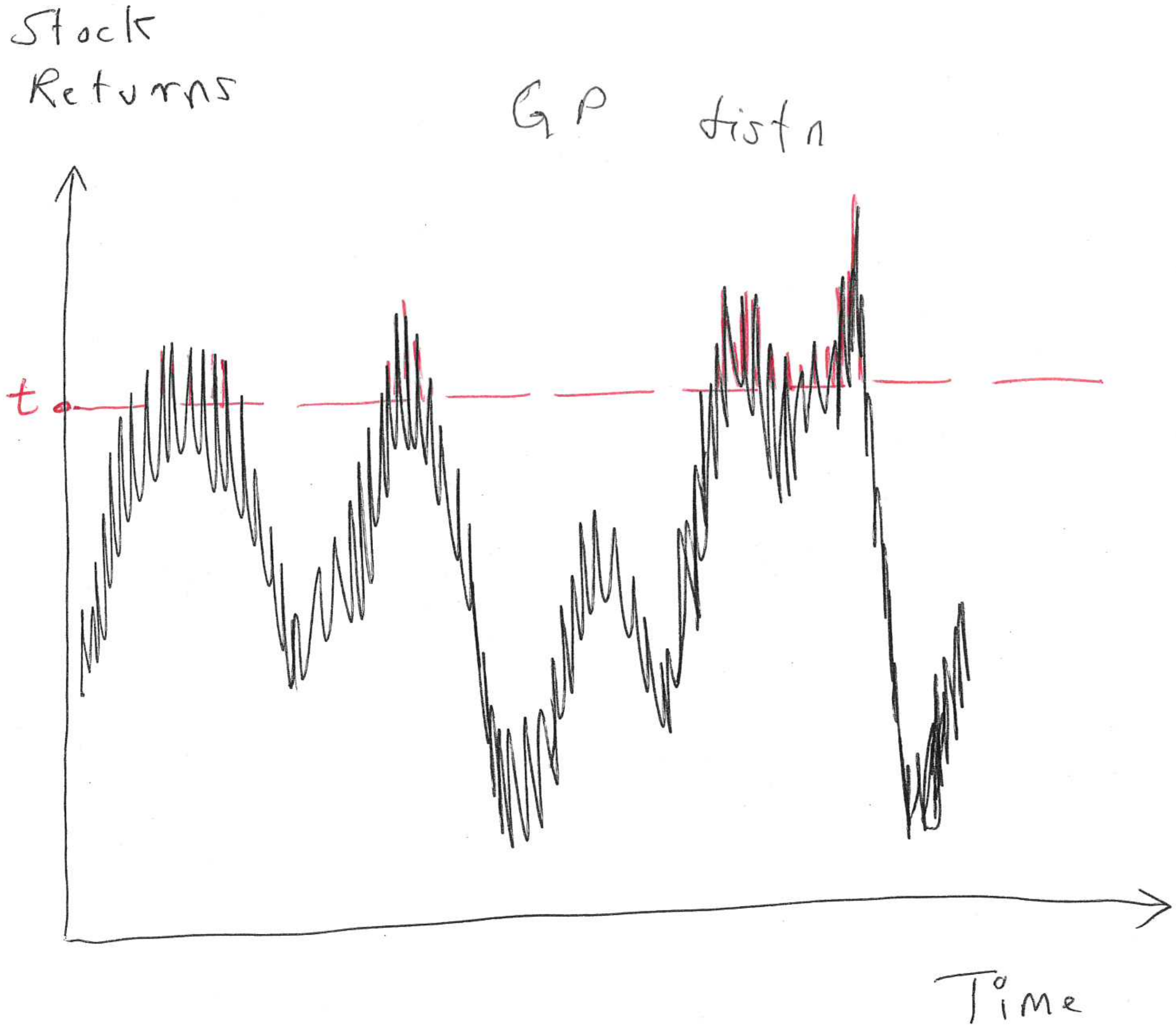
MLE

$$L(\sigma, \xi) = \prod_{i=1}^{n} \frac{w}{\sigma}\left[1+\xi\frac{X_i-t}{\sigma}\right]^{-\frac{1}{\xi}-1}$$

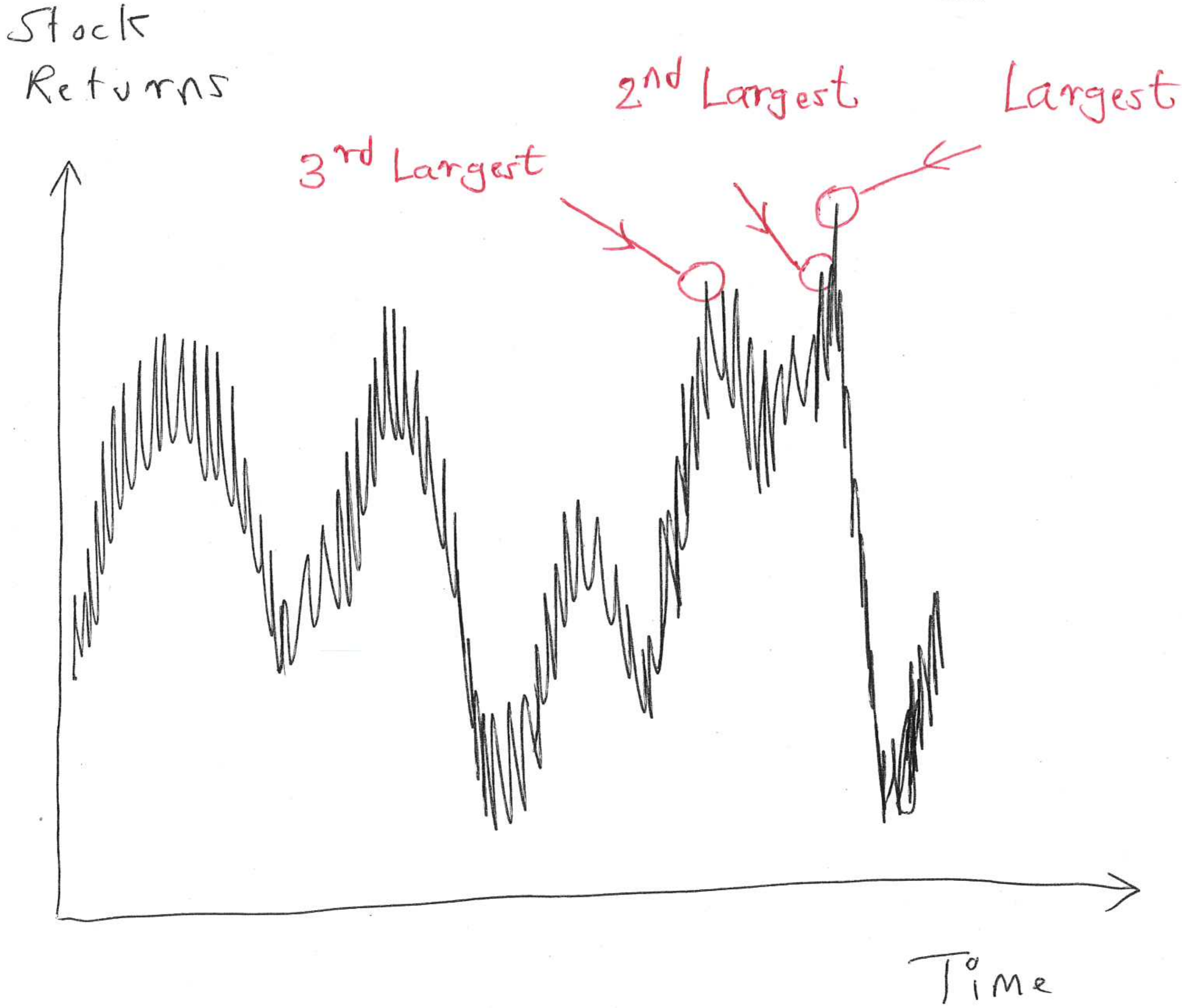
$$\log L = n\log w - n\log\sigma - \left(\frac{1}{\xi}+1\right)\sum_{i=1}^{n}\log\left(1+\xi\frac{X_i-t}{\sigma}\right) \qquad (*)$$

MLEs $\hat{\sigma}$ & $\hat{\xi}$ can be obtained by maximising $(*)$ with respect to σ & ξ. Use <u>optim</u> OR <u>nlm</u> in R.


GEV distn
Stock Returns
Largest
Time


Stock Returns
GP distn
t
Time

r- Largest Method



r - Largest Method

$M_n^{(1)}$ - Largest

$M_n^{(2)}$ - 2^{nd} Largest

$M_n^{(3)}$ - 3^{rd} Largest

$\vdots$

$M_n^{(r)}$ - r th Largest

According to Leadbetter (1983),

$$\Pr\left[\frac{M_n^{(1)}-b_n}{a_n} < x_1, \ldots, \frac{M_n^{(r)}-b_n}{a_n} < x_r\right]$$

$$\xrightarrow{n \to \infty} \sum_{s_1=0}^{1} \cdots \sum_{s_{r-1}=0}^{r-1-s_1 \ldots -s_{r-2}} \frac{(\gamma_2-\gamma_1)^{s_1}}{s_1!} \cdots \frac{(\gamma_r-\gamma_{r-1})^{s_{r-1}}}{s_{r-1}!} \cdot e^{-\gamma_r}$$

where $\gamma_i = -\log$ [CDF of GEV at x_i with $\mu = 0$ & $\sigma = 1$]

In practice, suppose

$$\left(M_{n,i}^{(1)}, M_{n,i}^{(2)}, \ldots, M_{n,i}^{(r)}\right),$$

$$i = 1, \ldots, m$$

are observations on the r largest values.

The joint PDF of $\left(M_n^{(1)}, \ldots, M_n^{(r)}\right)$ can be expressed as

$$\sigma^{-r} \, e^{-\left(1+\xi \frac{M_n^{(r)}-\mu}{\sigma}\right)^{-\frac{1}{\xi}}}$$

$$e^{-\left(\frac{1}{\xi}+1\right) \sum_{i=1}^{r} \log\left(1+\xi \frac{M_n^{(i)}-\mu}{\sigma}\right)}$$

How to estimate μ, σ & ξ?

MLE

$$L(\mu,\sigma,\xi) = \prod_{i=1}^{m} \left[\sigma^{-r} e^{-\left(1+\xi \frac{M_{n,i}^{(r)}-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} \cdot e^{-\left(\frac{1}{\xi}+1\right)\sum_{j=1}^{r} \log\left(1+\xi \cdot \frac{M_{n,i}^{(j)}-\mu}{\sigma}\right)} \right]$$

$$= \sigma^{-mr} e^{-\sum_{i=1}^{m}\left(1+\xi \frac{M_{n,i}^{(r)}-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} \cdot e^{-\left(\frac{1}{\xi}+1\right)\sum_{i=1}^{m}\sum_{j=1}^{r} \log\left(1+\xi \frac{M_{n,i}^{(j)}-\mu}{\sigma}\right)}$$

$$\log L(\mu, \sigma, \xi)$$

$$= \left(-mr \log \sigma\right) - \sum_{i=1}^{m} \left(1 + \xi \frac{M_{n,i}^{(r)} - \mu}{\sigma}\right)^{-\frac{1}{\xi}}$$

$$- \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{m} \sum_{j=1}^{r} \log\left(1 + \xi \frac{M_{n,i}^{(j)} - \mu}{\sigma}\right) \qquad (*)$$

Maximise $(*)$ with respect to (μ, σ, ξ) to obtain the MLEs $(\hat{\mu}, \hat{\sigma}, \hat{\xi})$. In R, this is can be done using optim or nlm

Estimation of GEV

$$\frac{\partial \log L}{\partial \mu} = -\frac{1}{\sigma}\sum_{i=1}^{n}\left(1+\xi\,\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1}$$

$$+\frac{\xi+1}{\sigma}\sum_{i=1}^{n}\left(1+\xi\,\frac{x_i-\mu}{\sigma}\right)^{-1}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} - \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\,\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}-1}$$

$$+\frac{\xi+1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)\left(1+\xi\,\frac{x_i-\mu}{\sigma}\right)^{-1}$$

$$\frac{\partial \log L}{\partial \xi} = -\frac{1}{\xi^2}\sum_{i=1}^{n}\left(1+\xi\,\frac{x_i-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\left[\log\left(1+\xi\,\frac{x_i-\mu}{\sigma}\right)\right.$$

$$\left.-\,\xi\left(1+\xi\,\frac{x_i-\mu}{\sigma}\right)^{-1}\frac{x_i-\mu}{\sigma}\right]$$

$$+\frac{1}{\xi^2}\sum_{i=1}^{n}\log\left(1+\xi\,\frac{x_i-\mu}{\sigma}\right)$$

$$-\frac{1+\xi}{\xi}\sum_{i=1}^{n}\left(1+\xi\,\frac{x_i-\mu}{\sigma}\right)^{-1}\frac{x_i-\mu}{\sigma}$$

The MLES $(\hat{\mu}, \hat{\sigma}, \hat{\xi})$ are the simultaneous solns of $\frac{\partial \log L}{\partial \mu} = 0$, $\frac{\partial \log L}{\partial \sigma} = 0$, $\frac{\partial \log L}{\partial \xi} = 0$. A quasi-Newton algorithm can be used to solve these eqns.

Estimation of GP

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \xi\left(1+\frac{1}{\xi}\right)\sum_{i=1}^{n}\left(1+\xi\frac{x_i-t}{\sigma}\right)^{-1}\frac{x_i-t}{\sigma^2}$$

$$\frac{\partial \log L}{\partial \xi} = \frac{1}{\xi^2}\sum_{i=1}^{n}\log\left(1+\xi\frac{x_i-t}{\sigma}\right)$$

$$-\left(1+\frac{1}{\xi}\right)\sum_{i=1}^{n}\left(1+\xi\frac{x_i-t}{\sigma}\right)^{-1}\frac{x_i-t}{\sigma}$$

The MLEs $\hat{\sigma}$ & $\hat{\xi}$ are the simultaneous soln of

$$\frac{\partial \log L}{\partial \sigma} = 0, \quad \frac{\partial \log L}{\partial \xi} = 0.$$

A quasi-Newton algorithm can be used to solve these eqns.

Sheet 4 Q1

$$L(\sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$

$$= \frac{1}{\sigma^n} e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} e^{-\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -n \log \sigma - \frac{1}{\sigma}\sum_{i=1}^{n} x_i$$

$$- \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}$$

$$\frac{d \log L(\sigma)}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i$$

$$- \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}} \frac{x_i}{\sigma^2} = 0$$

The MLE $\hat{\sigma}$ is the root of

$$-n\sigma + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}} = 0.$$

$$\frac{\partial}{\partial \lambda} \sum_{i=1}^{n} \sigma^{+\lambda} x_i^{-\lambda}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial \lambda} \left[\left(\frac{\sigma}{x_i} \right)^{\lambda} \right]$$

$$= \sum_{i=1}^{n} \left(\frac{\sigma}{x_i} \right)^{\lambda} \log \left(\frac{\sigma}{x_i} \right)$$

because

$$\boxed{\frac{\partial}{\partial \lambda} a^{\lambda} = a^{\lambda} \log a}$$

Sheet 4 Q2

$\cancel{L(\lambda)}$

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{\lambda} x_i^{-\lambda-1} e^{-\sigma^{\lambda} x_i^{-\lambda}} \right]$$

$$= \lambda^n \sigma^{n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{-\lambda-1} e^{-\sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda}}$$

$$\log L(\lambda, \sigma) = n \log \lambda + n\lambda \log \sigma$$

$$- (\lambda+1) \sum_{i=1}^{n} \log x_i - \sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda} \log\left(\frac{\sigma}{x_i}\right)$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \lambda \sigma^{\lambda-1} \sum_{i=1}^{n} x_i^{-\lambda}$$

The MLEs $\hat{\sigma}$ & $\hat{\lambda}$ are the simultaneous solns of $\frac{\partial \log L}{\partial \lambda} = 0$ & $\frac{\partial \log L}{\partial \sigma} = 0$.

Sheet 4 Q3

$$L(\lambda,\sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{-\lambda} x_i^{\lambda-1} e^{-\left(\frac{x_i}{\sigma}\right)^{\lambda}} \right]$$

$$= \lambda^n \sigma^{-n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{\lambda-1} \cdot e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}}$$

$$\log L = n \log \lambda - n\lambda \log \sigma + (\lambda-1)\sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda} \log\left(\frac{x_i}{\sigma}\right)$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \lambda \sigma^{-\lambda-1} \sum_{i=1}^{n} x_i^{\lambda}$$

The MLEs $\hat{\sigma}$ & $\hat{\lambda}$ are the simultaneous solns of $\frac{\partial \log L}{\partial \lambda} = 0$ & $\frac{\partial \log L}{\partial \sigma} = 0$.

Sheet 4, Q4

$$L(\lambda) = \prod_{i=1}^{n} \left[(1-\lambda x_i)^{\frac{1}{\lambda}-1} \right]$$

$$= \left[\prod_{i=1}^{n} (1-\lambda x_i) \right]^{\frac{1}{\lambda}-1}$$

$$\log L(\lambda) = \left(\frac{1}{\lambda}-1\right) \sum_{i=1}^{n} \log (1-\lambda x_i)$$

$$\frac{d}{d\lambda} (\log L(\lambda)) = -\frac{1}{\lambda^2} \sum_{i=1}^{n} \log (1-\lambda x_i)$$

$$+ \frac{1-\lambda}{\lambda} \sum_{i=1}^{n} \frac{(-x_i)}{1-\lambda x_i} = 0$$

The MLE $\hat{\lambda}$ is the root of

$$\sum_{i=1}^{n} \log (1-\lambda x_i) + \lambda (1-\lambda) \sum_{i=1}^{n} \frac{x_i}{1-\lambda x_i} = 0.$$

Portfolio

A collection of investments

X_1 = loss on investment 1

X_2 = loss on investment 2

$\vdots$

X_m = loss on investment m

- $X_1, X_2, \ldots, X_m$ are RVs
- m can be fixed OR itself a RV
- $X_1, X_2, \ldots, X_m$ could be IID

  OR dependent RVs

Variables of interest:

- $X_1 + \dots + X_m$ = total portfolio loss

- $a_1 X_1 + \dots + a_m X_m$ = weighted portfolio loss

e.g $a_m = \frac{1}{m}$

$$\Rightarrow \frac{X_1 + \dots + X_m}{m}$$

- $\max(X_1, X_2, \dots, X_m)$ largest loss

- $\min(X_1, X_2, \dots, X_m)$ smallest loss

- $\text{median}(X_1, X_2, \dots, X_m)$ median of loss

~~What~~ How to model $X_1, \ldots, X_m$?

- Gumbel distn

- Fréchet "

- Weibull "

- Pareto "

"Heavy tailed"

1) Gumbel distn (General Form)

$$F(x) = e^{-e^{-\frac{X-\mu}{\sigma}}}$$

CDF

$$f(x) = \frac{1}{\sigma} e^{-\frac{X-\mu}{\sigma}} e^{-e^{-\frac{X-\mu}{\sigma}}}$$

PDF

$$-\infty < x < \infty$$

$-\infty < \mu < \infty$ "location"

$\sigma > 0$ "scale"

Suppose $Y_1, \dots, Y_n$ are IID losses following this distn. What are the estimates of μ & σ?

$$L(\mu, \sigma) = \sigma^{-n} e^{-\sum_{i=1}^{n} \frac{Y_i-\mu}{\sigma} - \sum_{i=1}^{n} e^{-\frac{Y_i-\mu}{\sigma}}}$$

$$\log L = -n \log \sigma - \sum_{i=1}^{n} \frac{Y_i-\mu}{\sigma} - \sum_{i=1}^{n} e^{\frac{Y_i-\mu}{\sigma}}$$

$\hat{\mu}$ & $\hat{\sigma}$ are simult soln of

$$\left.\begin{aligned} \frac{\partial \log L}{\partial \mu} &= 0 \\ \frac{\partial \log L}{\partial \sigma} &= 0 \end{aligned}\right\}$$

$$\Leftrightarrow \begin{cases} \hat{\sigma} + \dfrac{\sum_{i=1}^{n} Y_i e^{-\frac{Y_i}{\hat{\sigma}}}}{\sum_{i=1}^{n} e^{-\frac{Y_i}{\hat{\sigma}}}} = \dfrac{1}{n} \sum_{i=1}^{n} Y_i \\ \hat{\mu} = -\hat{\sigma} \log \left\{ \dfrac{1}{n} \sum_{i=1}^{n} e^{-\frac{Y_i}{\hat{\sigma}}} \right\} \end{cases}$$

2) <u>Fréchet distn (general case)</u>

$$f(x) = \frac{\lambda}{\sigma}\left(\frac{\sigma}{x-\mu}\right)^{\lambda+1} e^{-\left(\frac{\sigma}{x-\mu}\right)^{\lambda}} \quad \text{PDF}$$

$$F(x) = e^{-\left(\frac{\sigma}{x-\mu}\right)^{\lambda}} \quad \text{CDF}$$

$-\infty < \mu < \infty$ "location"

$\sigma > 0$ "scale"

$\lambda > 0$ "shape"

$x > \mu$

Suppose $\cancel{X_1}, \ldots, \cancel{X_n}$ $y_1, \ldots, y_n$ are IID (loss) data from this distn. What are the estimates of $\hat{\mu}, \hat{\sigma}$ & $\hat{\lambda}$?

The log-likelihood is

$$\log L(\mu, \sigma, \lambda) = n \log \lambda + n\lambda \log \sigma - (\lambda+1) \sum_{i=1}^{n} \log (y_i - \mu) - \sigma^{\lambda} \sum_{i=1}^{n} (y_i - \mu)^{-\lambda}$$

$\hat{\mu}$ & $\hat{\sigma}$ & $\hat{\lambda}$ are simult solns of

$$\frac{\partial \log L}{\partial \mu} = 0, \quad \frac{\partial \log L}{\partial \sigma} = 0 \quad \& \quad \frac{\partial \log L}{\partial \lambda} = 0$$

$$\Longleftrightarrow \quad \frac{n}{\hat{\lambda}} + \frac{n \sum_{i=1}^{n} (y_i - \hat{\mu})^{-\lambda} \log(y_i - \hat{\mu})}{\sum_{i=1}^{n} (y_i - \hat{\mu})^{-\hat{\lambda}}} = \sum_{i=1}^{n} \log(y_i - \hat{\mu}),$$

$$\frac{n \hat{\lambda} \sum_{i=1}^{n} (y_i - \hat{\mu})^{-\hat{\lambda}-1}}{\sum_{i=1}^{n} (y_i - \hat{\mu})^{-\hat{\lambda}}} = (\hat{\lambda}+1) \sum_{i=1}^{n} \frac{1}{y_i - \hat{\mu}}$$

$$\hat{\sigma} = \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{\mu})^{-\hat{\lambda}} \right\}^{-\frac{1}{\hat{\lambda}}}$$

3) Weibull distn (general case)

$$f(x) = \frac{\lambda}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{\lambda-1} e^{-\left(\frac{x-\mu}{\sigma}\right)^{\lambda}} \qquad \text{PDF}$$

$$F(x) = 1 - e^{-\left(\frac{x-\mu}{\sigma}\right)^{\lambda}} \qquad \text{CDF}$$

$-\infty < \mu < \infty$ "location"

$\sigma > 0$ "scale"

$\lambda > 0$ "shape"

$x > \mu$

Suppose ~~X~~ $Y_1, \dots, Y_n$ loss data IID from this distn. What are the estimates of μ, σ & λ?

$$\log L(\mu, \sigma, \lambda) = n \log \lambda - n\lambda \log \sigma + (\lambda - 1)\sum_{i=1}^{n} \log(Y_i - \mu) - \sigma^{-\lambda} \sum_{i=1}^{n} (Y_i - \mu)^{\lambda}$$

$\hat{\mu}$, $\hat{\sigma}$ & $\hat{\lambda}$ simult solns of

$$\frac{\partial \log L}{\partial \mu} = 0, \quad \frac{\partial \log L}{\partial \sigma} = 0 \quad \& \quad \frac{\partial \log L}{\partial \lambda} = 0$$

$$\Leftrightarrow \frac{n}{\hat{\lambda}} + \sum_{i=1}^{n} \log(y_i - \hat{\mu}) = \frac{n \sum_{i=1}^{n} (y_i - \hat{\mu})^{\hat{\lambda}} \log(y_i - \hat{\mu})}{\sum_{i=1}^{n} (y_i - \hat{\mu})^{\hat{\lambda}}}$$

$$\frac{n\hat{\lambda} \sum_{i=1}^{n} (y_i - \hat{\mu})^{\hat{\lambda}-1}}{\sum_{i=1}^{n} (y_i - \hat{\mu})^{\hat{\lambda}}} = (\hat{\lambda} - 1) \sum_{i=1}^{n} \frac{1}{y_i - \hat{\mu}}$$

$$\hat{\sigma} = \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{\mu})^{\hat{\lambda}} \right\}^{\frac{1}{\hat{\lambda}}}$$

4) Pareto distn

$$f(x) = \frac{1}{\sigma}\left(1 - \frac{\lambda x}{\sigma}\right)^{\frac{1}{\lambda}-1} \qquad \text{PDF}$$

$$F(x) = 1 - \left(1 - \frac{\lambda x}{\sigma}\right)^{\frac{1}{\lambda}} \qquad \text{CDF}$$

Suppose $Y_1, \dots, Y_n$ are IID losses from this distn. What are the estimates of λ & σ?

$$\log L(\lambda, \sigma) = -n \log \sigma + \left(\frac{1}{\lambda} - 1\right) \sum_{i=1}^{n} \log\left(1 - \frac{\lambda Y_i}{\sigma}\right)$$

$\hat{\lambda}$ & $\hat{\sigma}$ are the simultaneous solns of

$$\frac{\partial \log L}{\partial \lambda} = 0 \quad \& \quad \frac{\partial \log L}{\partial \sigma} = 0$$

$$\Leftrightarrow \quad \frac{n}{\hat{\tau}} = \left(\frac{1}{\hat{\lambda}} - 1\right) \sum_{i=1}^{n} \frac{Y_i}{1 - \hat{\tau} Y_i},$$

$$\hat{\lambda} = -\frac{1}{n} \sum_{i=1}^{n} \log(1 - \hat{\tau} Y_i),$$

where $\hat{\tau} = \hat{\lambda} / \hat{\sigma}$.

Portfolio

A collection of investments

X_1 = loss on investment 1

X_2 = loss on investment 2

$\vdots$

X_m = loss on investment m

- $X_1, X_2, \ldots, X_m$ are RVs
- m can be fixed OR itself a RV
- $X_1, X_2, \ldots, X_m$ could be IID
  OR dependent RVs

Variables of interest:

- $X_1 + \dots + X_m$ = total portfolio loss

- $a_1 X_1 + \dots + a_m X_m$ = weighted portfolio loss

e.g $a_m = \frac{1}{m}$

$$\Rightarrow \frac{X_1 + \dots + X_m}{m}$$

- $\max(X_1, X_2, \dots, X_m)$ largest loss
- $\min(X_1, X_2, \dots, X_m)$ smallest loss
- $\text{median}(X_1, X_2, \dots, X_m)$ median of loss

i) $X_1, \dots, X_m$ IID & m fixed

ii) $X_1, \dots, X_m$ INID & m fixed

iii) $X_1, \dots, X_m$ dep RVs & m fixed

iv) $X_1, \dots, X_m$ IID & m RV

v) $X_1, \dots, X_m$ INID & m RV

vi) $X_1, \dots, X_m$ dep RV & m RV

1) Total Portfolio loss

i) $X_1, X_2, \ldots, X_m$ are IID & m fixed

$$T = X_1 + X_2 + \cdots + X_m$$

CDF $F_T(t) = P(T \leq t)$

$$= P(X_1 + X_2 + \cdots + X_m \leq t)$$

$$= \underset{X_1 + X_2 + \cdots + X_m \leq t}{\int\int \cdots \int} f(x_1)\, f(x_2) \cdots f(x_m)\, dx_m \cdots dx_2\, dx_1$$

where f denotes the PDF of $X_1, \ldots, X_m$.

PDF $f_T(t) = P(T = t)$

$$= \underset{X_1 + X_2 + \cdots + X_m = t}{\int\int \cdots \int} f(x_1)\, f(x_2) \cdots f(x_m)\, dx_m \cdots dx_2\, dx_1$$

mean $E(T) = E(X_1 + \cdots + X_m)$

$$= E(X_1) + \cdots + E(X_m)$$

$$= m E(X)$$

$$E(T^2) = E\left[(X_1 + \cdots + X_m)^2\right]$$

$$= E\left[X_1^2 + \cdots + X_m^2 + \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} X_i X_j\right]$$

$$= E(X_1^2) + \cdots + E(X_m^2) + \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} E(X_i X_j)$$

$$= m \cdot E(X^2) + \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} E(X_i) E(X_j)$$

$$= m E(X^2) + \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} (E(X))^2$$

$$= m E(X^2) + m(m-1)(E(X))^2$$

$$\mathrm{Var}(T) = E(T^2) - \left(E(T)\right)^2$$

$$= m\,E(X^2) + m(m-1)\left(E(X)\right)^2$$

$$- \left(m\,E(X)\right)^2$$

$$= m\,E(X^2) - m\left(E(X)\right)^2$$

$$= m\left[\,E(X^2) - \left(E(X)\right)^2\right]$$

$$= m \cdot \mathrm{Var}(X)$$

$$\boxed{\mathrm{Var}(T) = m \cdot \mathrm{Var}(X)}$$

ii) $X_1, X_2, \ldots, X_m$ INID & m fixed

INID "Independent and not identically distributed"

CDF

$$F_T(t) = P(T \leq t)$$

$$= P(X_1 + \cdots + X_m \leq t)$$

$$= \underset{x_1 + x_2 + \cdots + x_m \leq t}{\int\int \cdots \int} f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_m}(x_m) \, dx_m \cdots dx_2 \, dx_1$$

PDF

$$f_T(t) = P(T = t)$$

$$= P(X_1 + \cdots + X_m = t)$$

$$= \underset{x_1 + x_2 + \cdots + x_m = t}{\int\int \cdots \int} f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_m}(x_m) \, dx_m \cdots dx_2 \, dx_1$$

$$\text{mean } E(T) = E(X_1 + X_2 + \cdots + X_m)$$

$$= E(X_1) + E(X_2) + \cdots + E(X_m)$$

$$E(T^2) = E\left[(X_1 + X_2 + \cdots + X_m)^2\right]$$

$$= E\left[X_1^2 + X_2^2 + \cdots + X_m^2 + \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} X_i X_j\right]$$

$$= E(X_1^2) + E(X_2^2) + \cdots + E(X_m^2) + \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} E(X_i X_j)$$

$$= E(X_1^2) + E(X_2^2) + \cdots + E(X_m^2) + \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} E(X_i)\, E(X_j)$$

$$\mathrm{Var}(T) = E(T^2) - \left(E(T)\right)^2$$

$$= E(X_1^2) + \cdots + E(X_m^2)$$

$$+ \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} E(X_i)\, E(X_j)$$

$$- \left(E(X_1) + \cdots + E(X_m)\right)^2$$

$$= E(X_1^2) - \left(E(X_1)\right)^2$$

$$+ E(X_2^2) - \left(E(X_2)\right)^2$$

$$+ \cdots + E(X_m^2) - \left(E(X_m)\right)^2$$

$$\boxed{\mathrm{Var}(T) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_m)}$$

iii) $X_1, X_2, \ldots, X_m$ are dep RVs & m fixed

CDF

$$F_T(t) = P(T < t)$$

$$= P(X_1 + \cdots + X_m < t)$$

$$= \int\int_{X_1+X_2+\cdots+X_m<t} \cdots \int f(x_1, x_2, \ldots, x_m)\, dx_m \cdots dx_2\, dx_1$$

where f denotes the joint PDF of $(X_1, X_2, \ldots, X_m)$.

PDF

$$f_T(t) = \int\int_{X_1+X_2+\cdots+X_m=t} \cdots \int f(x_1, x_2, \ldots, x_m)\, dx_m \cdots dx_2\, dx_1$$

Mean

$$E(T) = E(X_1 + \cdots + X_m)$$

$$= E(X_1) + \cdots + E(X_m)$$

$$E(T^2) = E\left[(X_1 + \cdots + X_m)^2\right]$$

$$= E\left[X_1^2 + \cdots + X_m^2 + \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} X_i X_j\right]$$

$$= E(X_1^2) + \cdots + E(X_m^2) + \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} E(X_i X_j)$$

$$Var(T) = E(T^2) - (E(T))^2$$

$$= E(X_1^2) + \cdots + E(X_m^2) + \sum_{\substack{i=1 \\ i \neq j}}^{m} \sum_{j=1}^{m} E(X_i X_j) - (E(X_1) + \cdots + E(X_m))^2$$

iv) $X_1, X_2, \dots, X_M$ are IID & M is a RV

CDF $\quad F_T(t) = P(T < t)$

total prob rule

$$= P(X_1 + \cdots + X_M < t)$$

$$= \sum_{m=1}^{\infty} P(X_1 + \cdots + X_m < t \mid M = m)\, P(M = m)$$

$$= \sum_{m=1}^{\infty} P(X_1 + \cdots + X_m < t)\, P(M = m)$$

$$= \sum_{m=1}^{\infty} \left[\int \int \cdots \int_{X_1 + \cdots + X_m < t} f(x_1) \cdots f(x_m)\, dx_m \cdots dx_1 \right] \cdot P(M = m)$$

PDF

$$f_T(t) = \sum_{m=1}^{\infty} \left[\int \cdots \int_{X_1 + \cdots + X_m = t} f(x_1) \cdots f(x_m)\, dx_m \cdots dx_1 \right] \cdot P(M = m)$$

$$\boxed{E(X) = E[E(X|Y)]}$$

Mean

$$E(T) = E(X_1 + \cdots + X_M)$$

$$= E[E[(X_1 + \cdots + X_M) \mid M]]$$

$$= E[E(X_1) + \cdots + E(X_M)]$$

$$= E[M \cdot E(X)]$$

$$= E(M)\, E(X)$$

Variance

Homework

Find $Var(T)$?

Sheet 7

$$X = \max(X_1, X_2, \ldots, X_\alpha)$$

α fixed

$X_1, \ldots, X_\alpha$ IID $\mathrm{Exp}(\lambda)$

CDF $F_X(x) = P(X < x)$

$$= P[\max(X_1, \ldots, X_\alpha) < x]$$

$$= P[X_1 < x, \ldots, X_\alpha < x]$$

indep

$$= P(X_1 < x) \cdots P(X_\alpha < x)$$

$$= (1 - e^{-\lambda x}) \cdots (1 - e^{-\lambda x})$$

$$= (1 - e^{-\lambda x})^\alpha$$

2.

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$= \frac{d}{dx} \cdot \left(1-e^{-\lambda x}\right)^{\alpha}$$

$$= \lambda \alpha e^{-\lambda x} \left(1-e^{-\lambda x}\right)^{\alpha-1}$$

3. $$E(X^n) = \int_0^{\infty} x^n \lambda \alpha e^{-\lambda x} \left(1-e^{-\lambda x}\right)^{\alpha-1} dx$$

Set $y = e^{-\lambda x}$

$$x = -\frac{1}{\lambda} \log y$$

$$\frac{dx}{dy} = -\frac{1}{\lambda y}$$

$$= \int_1^0 \left(-\frac{1}{\lambda} \log y\right)^n \cancel{\lambda} \alpha \cancel{y} (1-y)^{\alpha-1} \left(-\frac{dy}{\cancel{\lambda} \cancel{y}}\right)$$

$$= -\alpha \int_1^0 \left(-\frac{1}{\lambda} \log y\right)^n (1-y)^{\alpha-1} dy$$

$$= \alpha \int_0^1 \frac{(-1)^n}{\lambda^n} (\log y)^n (1-y)^{\alpha-1}\, dy$$

$$= \frac{\alpha(-1)^n}{\lambda^n} \int_0^1 \underline{(\log y)}^n (1-y)^{\alpha-1}\, dy$$

$$= \frac{\alpha(-1)^n}{\lambda^n} \int_0^1 \frac{d^n}{da^n} y^a \Big|_{a=0} (1-y)^{\alpha-1}\, dy$$

$$= \frac{\alpha(-1)^n}{\lambda^n} \frac{d^n}{da^n} \left[\int_0^1 y^a (1-y)^{\alpha-1}\, dy \right] \Bigg|_{a=0}$$

$$\boxed{= \frac{\alpha(-1)^n}{\lambda^n} \frac{d^n}{da^n} B(a+1, \alpha) \Bigg|_{a=0}}$$

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1}\, dt$$

DEFN OF BETA FUN

$$\frac{d}{da} y^a = y^a \log y$$

$$\frac{d^2}{da^2} y^a = y^a (\log y)^2$$

$$\frac{d^3}{da^3} y^a = y^a (\log y)^3$$

$$\vdots$$

$$\frac{d^n}{da^n} y^a = y^a (\log y)^n$$

$$\Rightarrow \frac{d^n}{da^n} y^a \Big|_{a=0} = (\log y)^n$$

mean $E(X) = -\frac{\alpha}{\lambda} \frac{d}{da} B(a+1, \alpha) \Big|_{a=0}$

Var $Var(X) = \frac{\alpha}{\lambda^2} \frac{d^2}{da^2} B(a+1, \alpha) \Big|_{a=0}$

$$- \left[\frac{\alpha}{\lambda} \frac{d}{da} B(a+1, \alpha) \Big|_{a=0} \right]^2$$

8. Find the MLEs of λ & α.
(Suppose $\cancel{Y}_1, \ldots, \cancel{Y}_n$ is a random sample on X)

$$L(\alpha, \lambda) = \prod_{i=1}^{n} \left[\lambda \alpha e^{-\lambda y_i} (1 - e^{-\lambda y_i})^{\alpha - 1} \right]$$

$$= (\lambda \alpha)^n e^{-\lambda \sum_{i=1}^{n} y_i} \left[\prod_{i=1}^{n} (1 - e^{-\lambda y_i}) \right]^{\alpha - 1}$$

$$\log L = n \log(\lambda \alpha) - \lambda \sum_{i=1}^{n} y_i + (\alpha - 1) \sum_{i=1}^{n} \log\left(1 - e^{-\lambda y_i}\right)$$

$$\left.\begin{aligned} \frac{\partial \log L}{\partial \lambda} &= 0 \\[2em] \frac{\partial \log L}{\partial \alpha} &= 0 \end{aligned}\right\}$$

$\Longleftrightarrow$ Solve these eqns for $\hat{\alpha}$ & $\hat{\lambda}$

$$\mathrm{Var}(T) = ?$$

when $X_1, \dots, X_m$ IID & m RV

$$\mathrm{Var}(T) = E(T^2) - (E(T))^2$$

$$= E(T^2) - (E(M)\,E(X))^2$$

$$E(T^2) = E\left(E\left((X_1 + \dots + X_M)^2 \mid M\right)\right)$$

$$= E\Big(E(X_1^2) + \dots + E(X_M^2) + \sum_{\substack{i=1 \\ i \neq j}}^{M} \sum_{j=1}^{M} E(X_i)\,E(X_j) \;\Big|\; M\Big)$$

$$= E\left(M\,E(X^2) + M(M-1)\,(E(X))^2 \mid M\right)$$

$$= E(M)\,E(X^2) + E(M(M-1))\,(E(X))^2$$

$$\boxed{\mathrm{Var}(T) = E(M)\,\mathrm{Var}(X) + \mathrm{Var}(M)\,(E(X))^2}$$

2) $U = \min(X_1, \dots, X_m)$ - smallest of the losses out of the m investments

ii) $X_1, \dots, X_m$ INID & m fixed

<u>CDF</u>

$$F_U(u) = P(U < u)$$

$$= 1 - P(U > u)$$

$$= 1 - P(\min(X_1, \dots, X_m) > u)$$

$$= 1 - P(X_1 > u, \dots, X_m > u)$$

$$= 1 - P(X_1 > u) \cdot \cdot P(X_m > u)$$

$$= 1 - (1 - P(X_1 < u)) \cdot \cdot (1 - P(X_m < u))$$

$$= 1 - (1 - F_{X_1}(u)) \cdot \cdot (1 - F_{X_m}(u))$$

$$= 1 - \prod_{i=1}^{m} (1 - F_{X_i}(u))$$

PDF

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \frac{d}{du}\left\{1 - \prod_{i=1}^{m} \left(1 - F_{X_i}(u)\right)\right\}$$

$$= -\frac{d}{du} \prod_{i=1}^{m} \left(1 - F_{X_i}(u)\right)$$

$$= +\sum_{i=1}^{m} f_{X_i}(u) \prod_{\substack{j=1 \\ j \neq i}}^{m} \left(1 - F_{X_j}(u)\right)$$

Product rule for differentiation.

Mean

$$E(u) = \int u \cdot f_U(u)\, du$$

$$= \sum_{i=1}^{m} \int u\, f_{X_i}(u) \prod_{\substack{j=1 \\ j \neq i}}^{m} \left(1 - F_{X_j}(u)\right) du$$

Variance

$$Var(u) = E(u^2) - \left(E(u)\right)^2$$

$$= \sum_{i=1}^{m} \int u^2 f_{X_i}(u) \prod_{\substack{j=1 \\ j \neq i}}^{m} \left(1 - F_{X_j}(u)\right) du - \left(E(u)\right)^2$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

iii) $X_1, \ldots, X_m$ are dep RVs & m fixed

<u>CDF</u>

$$F_U(u) = P(U < u)$$

$$= P(\min(X_1, \ldots, X_m) < u)$$

$$= P(X_1 < u \text{ or } X_2 < u \text{ or} \ldots \text{or } X_m < u)$$

$$= P(\{X_1 < u\} \cup \{X_2 < u\} \cup \ldots \cup \{X_m < u\})$$

$$= P(X_1 < u) + P(X_2 < u) + \ldots + P(X_m < u)$$

$$- \sum_{i \neq j} P(\{X_i < u\} \cap \{X_j < u\}$$

$$+ \sum_{i \neq j \neq k} P(\{X_i < u\} \cap \{X_j < u\} \cap \{X_k < u\})$$

$$\vdots$$

$$+ (-1)^{m+1} P(\{X_1 < u\} \cap \{X_2 < u\} \cap \ldots \cap \{X_m < u\})$$

$$F_U(u) = \sum_{i=1}^{m} F_{X_i}(u)$$

joint CDF of (X_i, X_j)

$$- \sum_{i \neq j} F_{X_i, X_j}(u, u)$$

joint CDF of (X_i, X_j, X_k)

$$+ \sum_{i \neq j \neq k} F_{X_i, X_j, X_k}(u, u, u)$$

$$\vdots$$

$$+ (-1)^{m+1} F_{X_1, X_2, \ldots, X_m}(u, u, \ldots, u)$$

joint CDF of $(X_1, \ldots, X_m)$

PDF

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{i=1}^{m} f_{X_i}(u)$$

$$- \sum_{i \neq j} \frac{\partial}{\partial u} F_{X_i, X_j}(u, u)$$

$$+ \sum_{i \neq j \neq k} \frac{\partial}{\partial u} F_{X_i, X_j, X_k}(u, u, u)$$

$$\vdots$$

$$+ (-1)^{m+1} \frac{\partial}{\partial u} F_{X_1, \dots, X_m}(u, \dots, u).$$

3) $V = \max(X_1, \dots, X_m)$; maximum of the losses on the m investments.

ii) $X_1, X_2, \dots, X_m$ INID & m fixed

<u>CDF</u>

$$F_V(v) = P(V < v)$$

$$= P(\max(X_1, \dots, X_m) < v)$$

$$= P(X_1 < v, \dots, X_m < v)$$

$$= P(X_1 < v) \cdots P(X_m < v)$$

$$= F_{X_1}(v) \dots F_{X_m}(v)$$

$$= \prod_{i=1}^{m} F_{X_i}(v)$$

PDF

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= \frac{d}{dv} \prod_{i=1}^{m} F_{X_i}(v)$$

$$= \sum_{i=1}^{m} f_{X_i}(v) \prod_{\substack{j=1 \\ j \neq i}}^{m} F_{X_j}(v)$$

Product rule for differentiation.

Mean $\quad E(V) = \int v \cdot f_V(v)\, dv$

$$= \sum_{i=1}^{m} \int v\, f_{X_i}(v) \prod_{\substack{j=1 \\ j \neq i}}^{m} F_{X_j}(v)\, dv$$

Variance

$$\mathrm{Var}(V) = E(V^2) - (E(V))^2$$

$$= \sum_{i=1}^{m} \int v^2 f_{X_i}(v) \prod_{\substack{j=1 \\ j \neq i}}^{m} F_{X_j}(v)\, dv - (E(V))^2$$

iii) $X_1, \ldots, X_m$ dep RVs & m fixed

CDF $F_V(v) = P(V < v)$

$$= P(\max(X_1, \ldots, X_m) < v)$$

$$= P(X_1 < v, \ldots, X_m < v)$$

$$= F_{X_1, \ldots, X_m}(v, \ldots, v)$$

Joint CDF of $(X_1, \ldots, X_m)$

PDF

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= \frac{d}{dv} F_{X_1, \ldots, X_m}(v, \ldots, v)$$

Mean

$$E(V) = \int v \cdot f_V(v)\, dv$$

$$= \int v \cdot \frac{d}{dv} F_{X_1, \dots, X_m}(v, \dots, v)\, dv$$

Variance

$$Var(V) = E(V^2) - (E(V))^2$$

$$= \int v^2 \cdot \frac{d}{dv} F_{X_1, \dots, X_m}(v, \dots, v)\, dv - (E(V))^2$$

Financial Risk Measures

$X =$ Loss

$$P(X > £1 \text{ million}) > 0.9$$

$\Rightarrow$ you are in trouble

$$P(X > £1 \text{ million}) < 10^{-6}$$

$\Rightarrow$ ok to start a business

Let F denote the CDF of X.

$$F^{\leftarrow}(1-0.9) > £1 \text{ million}$$

$\Rightarrow$ you are in trouble

$$F^{\leftarrow}\cancel{(10^{-6})}(1-10^{-6}) < £1 \text{ million}$$

$\Rightarrow$ ok to start a business

X = Loss

What is the expected loss when there is one?

$$E[X \mid X > £1 \text{ million}]$$

---

1) Value at Risk

Defn: $VaR_p(X)$

$$= \inf\{u : F(u) \geq p\}$$

2) Expected Shortfall

$$ES_p(X) = \frac{1}{p}\left[E\left(X I\{X \leq VaR_p(X)\}\right) + p\, VaR_p(X) - VaR_p(X) \Pr(X \leq VaR_p(X))\right]$$

$$I\{A\} = \begin{cases} 1 & A \text{ is true} \\ 0 & A \text{ is false} \end{cases}$$

VaR was introduced by J P Morgan in 1960s

ES was introduced much recently in 1990s.

If X is an absolutely continuous RV then

$$\mathrm{VaR}_p(X) = F^{\leftarrow}(p)$$

and $$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_t(X)\,dt$$

1) Parametric methods

2) Non-parametric methods

1) Parametric methods

i) $X \sim N(\mu, \sigma^2)$

$$F(x) = \Phi\left(\frac{X-\mu}{\sigma}\right), \quad \Phi(\cdot) \text{ CDF of } N(0,1)$$

$$\Rightarrow F^{-1}(x) = \mu + \sigma\,\Phi^{-1}(x)$$

$$\Rightarrow \mathrm{VaR}_p(X) = \mu + \sigma\,\Phi^{-1}(p)$$

$$\Rightarrow \widehat{\mathrm{VaR}}_p(X) = \hat{\mu} + \hat{\sigma}\,\Phi^{-1}(p)$$

$\hat{\mu}$ is the MLE of μ

$\hat{\sigma}$ is the MLE of σ

If $X_1, \ldots, X_n$ is a random sample on X then

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X}$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\overline{X}_i - \overline{X})^2}$$

$$\Rightarrow \boxed{\widehat{VaR}_p(X) = \overline{X} + \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2} \cdot \Phi^{-1}(p)}$$

ii) $X \sim LN(\mu, \sigma^2)$

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right)$$

$$\Rightarrow F^{-1}(x) = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$\Rightarrow VaR_p(X) = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$\Rightarrow \widehat{VaR}_p(X) = e^{\hat{\mu} + \hat{\sigma} \Phi^{-1}(p)}$$

If $X_1, \dots, X_n$ is a random sample on X then

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log X_i$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\log X_i - \hat{\mu})^2}$$

$$\Rightarrow \boxed{\widehat{VaR}_p(X) = \exp\left[\frac{1}{n} \sum_{i=1}^{n} \log X_i + \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\log X_i - \hat{\mu})^2}\, \Phi^{-1}(p)\right]}$$

iii)

weights

$$X = \sum_{i=1}^{m} a_i X_i$$

portfolio loss

loss for ith investment

Suppose $X_i \sim N(\mu_i, \sigma_i^2)$ are independent.

Then $X \sim N\left(\sum_{i=1}^{m} a_i \mu_i, \sum_{i=1}^{m} a_i^2 \sigma_i^2\right)$

$$\Rightarrow VaR_p(X) = \sum_{i=1}^{m} a_i \mu_i + \sqrt{\sum_{i=1}^{m} a_i^2 \sigma_i^2}\, \Phi^{-1}(p)$$

$$\Rightarrow \widehat{VaR}_p(X) = \sum_{i=1}^{m} a_i \hat{\mu}_i + \sqrt{\sum_{i=1}^{m} a_i^2 \hat{\sigma}_i^2}\, \Phi^{-1}(p)$$

Let $X_{i,1}, X_{i,2}, \ldots, X_{i,n}$ be a random sample on X_i

$$\hat{\mu}_i = \frac{1}{n} \sum_{j=1}^{n} X_{i,j}$$

$$\hat{\sigma}_i = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(X_{i,j} - \hat{\mu}_i\right)^2}$$

Sheet 6, Q1

$$F(x) = 1 - e^{-\lambda x}$$

$$F^{\leftarrow}(x) = -\frac{1}{\lambda}\log(1-x)$$

$$VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = -\frac{1}{\lambda p}\int_0^p \log(1-t)\,dt$$

$$= -\frac{1}{\lambda p}\left\{\left[t \cdot \log(1-t)\right]_0^p + \int_0^p \frac{t}{1-t}\,dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) - 0 + \int_0^p \frac{t-1+1}{1-t}\,dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) + \int_0^p \left(-1 + \frac{1}{1-t}\right)dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) + \left[-t - \log(1-t)\right]_0^p\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) - p - \log(1-p)\right\}.$$

Sheet 6, Q2

$$F(x) = x^{a}$$

$$F^{\leftarrow}(x) = x^{\frac{1}{a}}$$

$$VaR_p(X) = p^{\frac{1}{a}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p t^{\frac{1}{a}}\,dt$$

$$= \frac{1}{p}\left[\frac{t^{\frac{1}{a}+1}}{\frac{1}{a}+1}\right]_0^p$$

$$= \frac{1}{p}\,\frac{p^{\frac{1}{a}+1}}{\frac{1}{a}+1}$$

$$= \frac{a}{p(a+1)}\,p^{\frac{1}{a}+1}$$

$$= \frac{a}{a+1}\,p^{\frac{1}{a}}$$

Sheet 6, Q3

$$F(x) = \frac{x-a}{b-a}$$

$$F^{\leftarrow}(x) = a + (b-a)x$$

$$VaR_p(X) = a + (b-a)p$$

$$ES_p(X) = \frac{1}{p}\int_0^p \left[a + (b-a)t\right] dt$$

$$= \frac{1}{p}\left[a\cancel{t} + (b-a)\frac{t^2}{2}\right]_0^p$$

$$= \frac{1}{p}\left[ap + (b-a)\frac{p^2}{2}\right]$$

$$= a + (b-a)\frac{p}{2}$$

Sheet 6, Q4

$$F(x) = 1 - \left(\frac{K}{x}\right)^{a}$$

$$F^{\leftarrow}(x) = K \, (1-x)^{-\frac{1}{a}}$$

$$VaR_p(X) = K(1-p)^{-\frac{1}{a}}$$

$$ES_p(X) = \frac{K}{p}\int_0^p (1-t)^{-\frac{1}{a}}\,dt$$

$$= \frac{K}{p}\left[\frac{(1-t)^{1-\frac{1}{a}}}{(-1)(1-\frac{1}{a})}\right]_0^p$$

$$= \frac{Ka}{p(1-a)}\left[(1-p)^{1-\frac{1}{a}} - 1\right]$$

Sheet 6, Q5

$$F(x) = \Phi(x)$$

$$F^{-1}(x) = \Phi^{-1}(x)$$

$$VaR_p(X) = \Phi^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \Phi^{-1}(t)\,dt$$

Sheet 6, Q6

$$F(x) = \left[1 + \left(\frac{x}{a}\right)^{-b}\right]^{-1}$$

$$F^{\leftarrow}(x) = a\left(\frac{1}{x} - 1\right)^{-\frac{1}{b}}$$

$$= a\, x^{\frac{1}{b}} (1-x)^{-\frac{1}{b}}$$

$$\mathrm{VaR}_p(X) = a\, p^{\frac{1}{b}} (1-p)^{-\frac{1}{b}}$$

$$\mathrm{ES}_p(X) = \frac{a}{p} \int_0^p t^{\frac{1}{b}} (1-t)^{-\frac{1}{b}}\, dt$$

$$= \frac{a}{p} B_p\left(\frac{1}{b} + 1, 1 - \frac{1}{b}\right)$$

where

$$B_x(a, b) = \int_0^x t^{a-1} (1-t)^{b-1}\, dt$$

"INCOMPLETE BETA FUNCTION"

Sheet 6, Q7

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}$$

$$F^{\leftarrow}(x) = \lambda\left[(1-x)^{-\frac{1}{\alpha}} - 1\right]$$

$$VaR_p(X) = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right]$$

$$ES_p(X) = \frac{\lambda}{p}\int_0^p \left[(1-t)^{-\frac{1}{\alpha}} - 1\right] dt$$

$$= \frac{\lambda}{p}\left[\frac{(1-t)^{1-\frac{1}{\alpha}}}{(-1)\left(1-\frac{1}{\alpha}\right)} - t\right]_0^p$$

$$= \frac{\lambda}{p}\left[\frac{\alpha(1-p)^{1-\frac{1}{\alpha}}}{(1-\alpha)} - p + \frac{\alpha}{\alpha-1}\right]$$

Financial Risk Measures

X = Loss with CDF F

If X is absolutely cont RV then

$\underline{VaR_p(X)} = F^{\leftarrow}(p)$

"loss associated with probability p"

eg $VaR_{0.9}(X) = £1$ million

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(x)\,dt$$

"Expected loss given there is one"

eg $ES_{0.9}(x) = £2$ million

$\Leftrightarrow E[X \mid X > VaR_{0.9}(X)] = £2$ million

Estimation methods for VaR

- Parametric methods
- Non-parametric methods

Parametric estimation methods

iv) $X_1, \ldots, X_n \sim \text{Uni}[a, b]$
(loss data)

$$F(x) = \frac{x-a}{b-a}$$

$$\Rightarrow F^{\leftarrow}(x) = a + (b-a)x$$

$$\Rightarrow VaR_p(X) = a + (b-a)x$$

$$\Rightarrow \widehat{VaR}_p(X) = \hat{a} + (\hat{b} - \hat{a})p$$

$\hat{a} = \min(X_1, \ldots, X_n)$, the MLE of a

$\hat{b} = \max(X_1, \ldots, X_n)$, the MLE of b

$$\Rightarrow \widehat{VaR}_p(X) = \min(X_1, \ldots, X_n) + \left[\max(X_1, \ldots, X_n) - \min(X_1, \ldots, X_n)\right]p$$

v) $X_1, \dots, X_n \sim \mathrm{Exp}(\lambda)$
(loss fata)

$$F(x) = 1 - e^{-\lambda x}$$

$$\Rightarrow F^{\leftarrow}(x) = -\frac{1}{\lambda} \log(1-x)$$

$$\Rightarrow \mathrm{VaR}_p(X) = -\frac{1}{\lambda} \log(1-p)$$

$$\Rightarrow \widehat{\mathrm{VaR}}_p(X) = -\frac{1}{\hat{\lambda}} \log(1-p)$$

the MLE of λ is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{X}}$$

$$\Rightarrow \widehat{\mathrm{VaR}}_p(X) = -\bar{X} \log(1-p)$$

2) Non-parametric estimation methods

$X_1, \ldots, X_n$ - assume no specific distribution for this data
(loss data)

i) Historical method

- sort the data : $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$
- $\widehat{VaR}_p(X) = X_{(i)}$ if $p \in \left(\frac{i-1}{n}, \frac{i}{n}\right]$

eg

$n = 5$

$-1, \; 2, \; 3, \; 8, \; 5$

-1	2	3	5	8
$X_{(1)}$	$X_{(2)}$	$X_{(3)}$	$X_{(4)}$	$X_{(5)}$

$\widehat{VaR}_{0.5}(X) = X_{(3)} = 3$

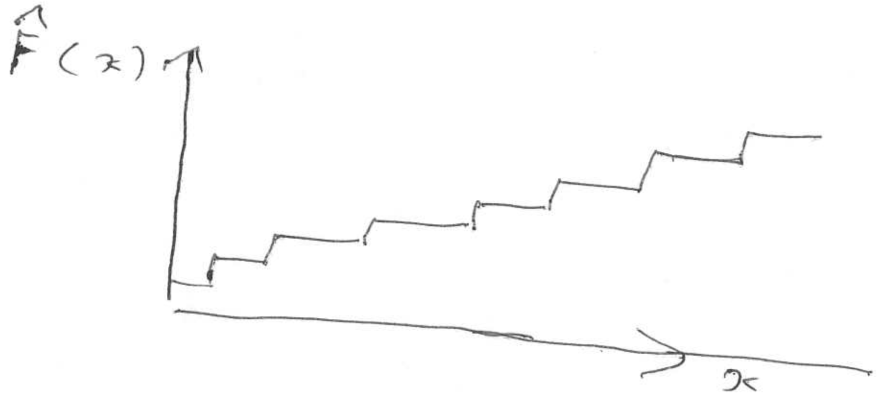
$\widehat{VaR}_{0.9}(X) = X_{(5)} = 8$

ii) <u>Bootstrap</u> <u>method</u>

Data: $X_1, X_2, \ldots, X_n$

- Let $\hat{F}$ denote empirical CDF of the data

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} I\{X_i \le x\}$$



- simulate B indep samples from $\hat{F}$
- Compute VaR using the <u>historical method</u> for each of the B samples
$\Rightarrow \widehat{VaR}_p^{(1)}, \widehat{VaR}_p^{(2)}, \ldots, \widehat{VaR}_p^{(B)}$
- $\widehat{VaR}_p(X) = \text{mean}\left(\widehat{VaR}_p^{(1)}, \ldots, \widehat{VaR}_p^{(B)}\right)$

(ii) <u>Jackknife</u> <u>method</u>

Data : $X_1, X_2, \ldots, X_n$

- compute VaR using <u>historical</u> <u>method</u> for $X_2, \ldots, X_n$, denote by $\widehat{VaR}_p^{(1)}$.

- compute VaR using <u>historical</u> <u>method</u> for $X_1, X_3, \ldots, X_n$, denote by $\widehat{VaR}_p^{(2)}$

⋮

- compute VaR using <u>historical</u> <u>method</u> for $X_1, X_2, \ldots, X_{n-1}$, denote by $\widehat{VaR}_p^{(n)}$

- $\widehat{VaR}_p(X) = \text{mean}\left(\widehat{VaR}_p^{(1)}, \widehat{VaR}_p^{(2)}, \ldots, \widehat{VaR}_p^{(n)}\right)$

iv) Kernel estimation method

Data: $X_1, X_2, \ldots, X_n$

The <u>Kernel</u> <u>CDF</u> is defined as

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} G\left(\frac{x - x_i}{h} \right)$$

kernel

bandwith

The estimate of $VaR_p(X)$ is given as

- the root of $\hat{F}(x) = p$

or
- $$\widehat{VaR}_p(X) = \frac{\sum_{i=1}^{n} \hat{F}\left(\frac{i - \frac{1}{2}}{n} - p \right) X_{(i)}}{\sum_{i=1}^{n} \hat{F}\left(\frac{i - \frac{1}{2}}{n} - p \right)}$$

v) Data: $X_1, \ldots, X_n$

Let $i = [np + \frac{1}{2}]$, $j = [np]$, $k = [(n+1)p]$, $g = np - j$, $h = (n+1)p - k$. Then VaR can be estimated by one of the following

$$\widehat{VaR}_p = (1-g) X_{(j)} + g X_{(j+1)}$$

$$\widehat{VaR}_p = \begin{cases} X_{(j)} & g < \frac{1}{2} \\ X_{(j+1)} & g \geq \frac{1}{2} \end{cases}$$

$$\widehat{VaR}_p = \begin{cases} X_{(j)} & g = 0 \\ X_{(j+1)} & g > 0 \end{cases}$$

$$\widehat{VaR}_p = (1-h) X_{(k)} + h X_{(k+1)}$$

$$\widehat{VaR}_p = \begin{cases} \dfrac{X_{(j)} + X_{(j+1)}}{2} & g = 0 \\ X_{(j+1)} & g > 0 \end{cases}$$

$$\widehat{VaR}_p = X_{(j+1)}$$

$$\widehat{VaR}_p = \left(\tfrac{1}{2} + i - np\right) X_{(i)} + \left(\tfrac{1}{2} - i + np\right) X_{(i+1)}$$

X = Loss with CDF F

$$VaR_p(X) = \inf\{u : F(u) \geq p\}$$

$$ES_p(X) = \frac{1}{p}\Big[E(X\,I\{X \leq VaR_p(X)\}) + p\,VaR_p(X) - VaR_p(X)\Pr(X \leq VaR_p(X))\Big]$$

If X is absolutely continous RV

$$VaR_p(X) = F^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

Mathematical Properties of VaR

a). $VaR_p(X+c) = VaR_p(X) + c$

"translation equivalent"

b). $VaR_p(cX) = c\, VaR_p(X), \quad c > 0$

"positively homogeneous"

c) $VaR_p(X) \equiv -VaR_{1-p}(-X)$

d). $X \geq 0 \Rightarrow VaR_p(X) \geq 0$

e) $X \geq Y \Rightarrow VaR_p(X) \geq VaR_p(Y)$

"monotonicity"

Mathematical properties of ES

a) $X \geq Y \Rightarrow ES_p(X) \geq ES_p(Y)$

b) $X \geq 0 \Rightarrow ES_p(X) \geq 0$

c) $ES_p(cX) = c \cdot ES_p(X) \quad , c > 0$

d) $ES_p(X + c) = ES_p(X) + c$

e) $ES_p(X + Y) \leq ES_p(X) + ES_p(Y)$

ii) Parametric methods

i) $X \sim N(\mu, \sigma^2)$

$$F^{-1}(p) = \mu + \sigma \Phi^{-1}(p)$$

$$ES_p(X) = \frac{1}{p} \int_0^p \left[\mu + \sigma \Phi^{-1}(t)\right] dt$$

$$= \mu + \frac{\sigma}{p} \int_0^p \Phi^{-1}(t)\, dt$$

Suppose $X_1, \ldots, X_n$ is a random sample on X. The MLEs of μ and σ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X},$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2}.$$

So, the MLE of $ES_p(X)$ is

$$\hat{\mu} + \frac{\hat{\sigma}}{p} \int_0^p \Phi^{-1}(t)\,dt$$

$$\boxed{= \bar{X} + \frac{1}{p}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} \int_0^p \Phi^{-1}(t)\,dt}$$

ii) $X \sim LN(\mu, \sigma^2)$

$$F^{-1}(p) = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$ES_p(X) = \frac{1}{p} \int_0^p e^{\mu + \sigma \Phi^{-1}(t)} \, dt$$

$$= \frac{e^{\mu}}{p} \int_0^p e^{\sigma \Phi^{-1}(t)} \, dt$$

Suppose $X_1, X_2, \ldots, X_n$ is a random sample on X. Then the MLEs of μ & σ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log X_i$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\log X_i - \hat{\mu})^2}$$

So, the MLE of $ES_p(X)$ is

$$\boxed{\frac{e^{\hat{\mu}}}{p} \int_0^p e^{\hat{\sigma} \Phi^{-1}(t)} \, dt}$$

iii) $X \sim \mathrm{Uni}[a, b]$

$$F^{-1}(p) = a + (b-a)p$$

$$ES_p(X) = \frac{1}{p}\int_0^p [a + (b-a)t]\,dt$$

$$= \frac{1}{p}\left[at + \frac{(b-a)t^2}{2}\right]_0^p$$

$$= a + \frac{(b-a)p}{2}$$

Suppose $X_1, \ldots, X_n$ is a random sample on X. Then the MLEs of a & b are

$$\hat{a} = \min(X_1, \ldots, X_n)$$
$$\hat{b} = \max(X_1, \ldots, X_n).$$

So, the MLE of $ES_p(X)$ is

$$\boxed{\min(X_1, \ldots, X_n) + \frac{p}{2}\left[\max(X_1, \ldots, X_n) - \min(X_1, \ldots, X_n)\right]}$$

Estimation methods for ES

- Parametric methods
- Nonparametric methods

2) Non-parametric methods for ES

i) Historical method

Data : $x_1, x_2, \dots, x_n$

Order the data: $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$

↑ smallest ↑ largest

$$\boxed{\widehat{ES}_p(X) = \frac{\sum_{i=[np]}^{n} X_{(i)}}{n - [np]}}$$

e.g

$-3 \quad 3 \quad 0 \quad -5 \quad 2$

$$\begin{array}{ccccc} -5 & -3 & 0 & 2 & 3 \\ \| & \| & \| & \| & \| \\ X_{(1)} & X_{(2)} & X_{(3)} & X_{(4)} & X_{(5)} \end{array}$$

$$\widehat{ES}_{0.6}(X) = \frac{0+2+3}{5-3} = \frac{5}{2} = 2.5$$

There are many variations of HM:

ii) due to Yamai & Yoshiba (2002)

$$\hat{ES}_p(X) = \frac{1}{n(\alpha-\beta)} \sum_{i=[n\beta]}^{[n\alpha]} X_{(i)}$$

α, β fixed consts depending on p.

iii) due to Inui & Kijima (2005)

$$\hat{ES}_p(X) = \begin{cases} -\bar{X}_{k:n} & \text{if } n(1-p) \text{ is an integer} \\ -p\bar{X}_{k:n} - (1-p)\bar{X}_{k+1:n} & \text{if } n(1-p) \text{ is not an integer} \end{cases}$$

where $\bar{X}_{k:n} = \frac{1}{k}\left[X_{(1)} + \cdots + X_{(k)}\right]$.

iv) due to Chen (2008)

$$\widehat{ES}_p(X) = \frac{1}{1+[np]} \sum_{i=1}^{n} X_i I\{X_i \geq X_{([n(1-p)]+1)}\}$$

v) due to Peraochi & Tanase (2008)

$$\widehat{ES}_p(X) = \frac{1}{np} \sum_{i=1}^{[np]} X_{(i)} + \left(1 - \frac{[np]}{np}\right) X_{([np]+1)}$$

Sheet 2, Q7

$$F(x) = 1 - q^{(x+1)^\alpha}$$

$$\lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)} = 0 \iff \text{ETT holds}$$

$$\boxed{P(X=k) = F(k) - F(k-1)}$$

TRUE for any discrete RV

$$\Rightarrow P(X=k) = 1 - q^{(k+1)^\alpha} - (1 - q^{k^\alpha})$$

$$= q^{k^\alpha} - q^{(k+1)^\alpha}$$

$$= q^{k^\alpha}\left[1 - q^{(k+1)^\alpha - k^\alpha}\right]$$

$$\Rightarrow \lim_{k\to\infty} \frac{P(X=k)}{1-F(k-1)} = \lim_{k\to\infty} \frac{q^{k^\alpha}\left[1 - q^{(k+1)^\alpha - k^\alpha}\right]}{1 - (1 - q^{k^\alpha})}$$

$$= \lim_{k\to\infty} \frac{q^{k^\alpha}\left[1 - q^{(k+1)^\alpha - k^\alpha}\right]}{q^{k^\alpha}}$$

$$= \lim_{k\to\infty} \left[1 - q^{(k+1)^\alpha - k^\alpha}\right]$$

$$= 0 \quad \text{if } \alpha < 1$$

$$(k+1)^{\alpha} - k^{\alpha}$$

$$= k^{\alpha}\left(1+\frac{1}{k}\right)^{\alpha} - k^{\alpha}$$

$$= k^{\alpha}\left[\left(1+\frac{1}{k}\right)^{\alpha} - 1\right]$$

$$= k^{\alpha}\left[\sum_{j=0}^{\infty}\binom{\alpha}{j}\left(\frac{1}{k}\right)^{j} - 1\right]$$

$$= k^{\alpha}\left[\cancel{1} + \frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{2}\cdot\frac{1}{k^2} + \cdots - \cancel{1}\right]$$

$$= \alpha k^{\alpha-1} + \frac{\alpha(\alpha-1)}{2}\cdot k^{\alpha-2} + \cdots,$$

Suppose $\alpha < 1$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0 \qquad \cdots$$

$$\Rightarrow (k+1)^{\alpha} - k^{\alpha} \to 0 \text{ as } k \to \infty$$

Sheet 2, Q4

$$p(k) = \frac{k^{-s}}{\zeta(s)}$$

$$1-F(k-1) = \sum_{i=k}^{\infty} p(i)$$

$$= \sum_{i=k}^{\infty} \frac{i^{-s}}{\zeta(s)}$$

$$= \frac{1}{\zeta(s)} \sum_{i=k}^{\infty} i^{-s}$$

$$\approx \frac{1}{\zeta(s)} \int_k^{\infty} x^{-s}\,dx$$

$$= \frac{1}{\zeta(s)} \left[\frac{x^{1-s}}{1-s}\right]_k^{\infty}$$

$$= \frac{1}{\zeta(s)} \left[0 - \frac{k^{1-s}}{1-s}\right], \quad s>1$$

$$= \frac{1}{\zeta(s)} \frac{k^{1-s}}{s-1}$$

$$\lim \frac{p(k)}{1-F(k-1)} = \lim \frac{s-1}{k} = 0 \Rightarrow \text{ETT holds}$$

$$F(x) = 1 - \left[1 - e^{-\frac{\lambda}{x}}\right]^{\alpha}, \quad w(F) = \infty$$

$$\mathrm{I}: \lim_{t\to\infty} \frac{1 - F(t + x\gamma(t))}{1 - F(t)}$$

$$= \lim_{t\to\infty} \frac{\left[1 - e^{-\frac{\lambda}{t + x\gamma(t)}}\right]^{\alpha}}{\left[1 - e^{-\frac{\lambda}{t}}\right]^{\alpha}}$$

$$= \lim_{t\to\infty} \left[\frac{1 - e^{-\frac{\lambda}{t + x\gamma(t)}}}{1 - e^{-\frac{\lambda}{t}}}\right]^{\alpha}$$

$\boxed{e^{-x} \approx 1 - x \text{ if } x \text{ is small}}$

$$\approx \lim_{t\to\infty} \left[\frac{\cancel{1} - \left(\cancel{1} - \frac{\cancel{\lambda}}{t + x\gamma(t)}\right)}{\cancel{1} - \left(\cancel{1} - \frac{\cancel{\lambda}}{t}\right)}\right]^{\alpha}$$

$$= \lim_{t\to\infty} \left[\frac{t}{t + x\gamma(t)}\right]^{\alpha}$$

$$= \lim_{t\to\infty} \left[\frac{1}{1 + x\frac{\gamma(t)}{t}}\right]^{\alpha} \neq e^{-x}$$

$\Rightarrow$ Cond (I) not satisfied

(II)

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)}$$

$$= \lim_{t \to \infty} \frac{\cancel{1} - \left\{ \cancel{1} - \left[1 - e^{-\frac{\lambda}{tx}} \right]^{\alpha} \right\}}{\cancel{1} - \left\{ \cancel{1} - \left[1 - e^{-\frac{\lambda}{t}} \right]^{\alpha} \right\}}$$

$$= \lim_{t \to \infty} \left[\frac{1 - e^{-\frac{\lambda}{tx}}}{1 - e^{-\frac{\lambda}{t}}} \right]^{\alpha}$$

$$\boxed{e^{-x} \approx x \quad \text{for smale } x}$$

$$\approx \lim_{t \to \infty} \left[\frac{\cancel{1} - \left(\cancel{1} - \frac{\lambda}{tx} \right)}{\cancel{1} - \left(\cancel{1} - \frac{\lambda}{t} \right)} \right]^{\alpha}$$

$$= \lim_{t \to \infty} \left(\frac{1}{x} \right)^{\alpha} = x^{-\alpha} \Rightarrow$$

Cond $\underline{\text{II}}$ satisfied.

Non-parametric Estimation of ES

i) Historical method

ii) variant due to Yamai & Yoshiba (2002)

iii) variant due to Inui & Kijima (2005)

iv) variant due to Chen (2008)

v) variant due to Peracchi & Tanase (2008)

vi) Kernel method

$$\widehat{ES}_p = \frac{1}{np} \sum_{i=1}^{n} X_i \, A_h\left(\hat{q}(p) - X_i\right)$$

where

$$\hat{q}(p) = \sum_{i=1}^{n} \left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(t-p)\,dt \right] X_{(i)},$$

"kernel"

$$K_h(u) = \frac{1}{h} K\left(\frac{u}{h}\right),$$

"band witdth"

$$A(x) = \int_{-\infty}^{x} K(u)\,du, \quad A_h(u) = A\left(\frac{u}{h}\right)$$

vii) Bootstrap estimator

Data : $X_1, X_2, \dots, X_n$

- construct the empirical CDF

$$\hat{F}(t) = \frac{1}{n} \sum_{i=1}^{n} I\{X_i \le t\}$$

- ~~cons~~ simulate B samples each of size n from $\hat{F}$

- use the historical method to estimate ES for the B samples, resulting in $\hat{ES}_p^{(1)}, \hat{ES}_p^{(2)}, \dots, ES_p^{(B)}$

- $\hat{ES}_p = \text{mean}\left(\hat{ES}_p^{(1)}, ES_p^{(2)}, \dots, ES_p^{(B)}\right)$.

Bootstrap estimator $\hat{ES}_p$.

viii) Jackknife estimator

Data : $X_1, X_2, \dots, X_n$

- estimate ES by the historical method for $X_2, X_3, \dots, X_n$, resulting in $\hat{ES}_p^{(1)}$

- estimate ES by the historical method for $X_1, X_3, \dots, X_n$, resulting in $\hat{ES}_p^{(2)}$

⋮

- estimate ES by the historical method for $X_1, X_2, \dots, X_{n-1}$, resulting in $\hat{ES}_p^{(n)}$.

- $\hat{ES}_p = \text{mean}\left(\hat{ES}_p^{(1)}, \hat{ES}_p^{(2)}, \dots, ES_p^{(n)}\right)$

Jackknife estimator of ES.

ix) Richardson's method

(Prof at School of Math, Uni. of Manchester long time ago.)

Data: $X_1, \ldots, X_n$

a) Compute the empirical cdf $\hat{F}(\cdot)$

b) Simulate $Y_1, \ldots, Y_N$ from $\hat{F}(\cdot)$

c) estimate ES by historical method for $Y_1, \ldots, Y_N$

d) Repeat b) and c) say 1000 times resulting in $\widehat{ES}_{N,1}, \widehat{ES}_{N,2}, \ldots, \widehat{ES}_{N,1000}$.

e) Set

$$m_N = \frac{1}{1000} \sum_{i=1}^{1000} \widehat{ES}_{N,i}$$

f) set $s_n = m_{N_n}$ for $n = 1, 2, \ldots, k+1$ for some $k, N_1, N_2, \ldots, N_{k+1}$

[e.g. $k = 2$, $N_1 = 100$, $N_2 = 200$, $N_3 = 300$]

eg $k = 2, N_1 = 100, N_2 = 200, N_3 = 300$

$$\hat{ES}_v = \frac{\begin{vmatrix} m_{100} & m_{200} & m_{300} \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{4} & \frac{1}{9} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{4} & \frac{1}{9} \end{vmatrix}}$$

$$= \frac{m_{100} \cdot \left(\frac{1}{18} - \frac{1}{12}\right) - m_{200}\left(\frac{1}{9} - \frac{1}{3}\right) + m_{300} \cdot \left(\frac{1}{4} - \frac{1}{2}\right)}{\frac{1}{18} - \frac{1}{12} - \left(\frac{1}{9} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{2}\right)}$$

g)

$$\hat{ES}_p = \frac{\begin{vmatrix} S_1 & S_2 & \cdots & S_{k+1} \\ 1 & \frac{1}{2} & \cdots & \frac{1}{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1^k & \left(\frac{1}{2}\right)^k & \cdots & \left(\frac{1}{k+1}\right)^k \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \frac{1}{2} & \cdots & \frac{1}{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1^k & \left(\frac{1}{2}\right)^k & \cdots & \left(\frac{1}{k+1}\right)^k \end{vmatrix}}$$

Richardson's estimator

$w(F) = \sup \{ x : F(x) < 1 \}$

if F is abs cont CDF

then take $w(F)$ as the x

s.t. $F(x) = 1$.

$$F(x) = 1 - q^{(x+1)^a} = 1$$

$$\Rightarrow q^{(x+1)^a} = 0 \quad , \quad 0 < q < 1$$

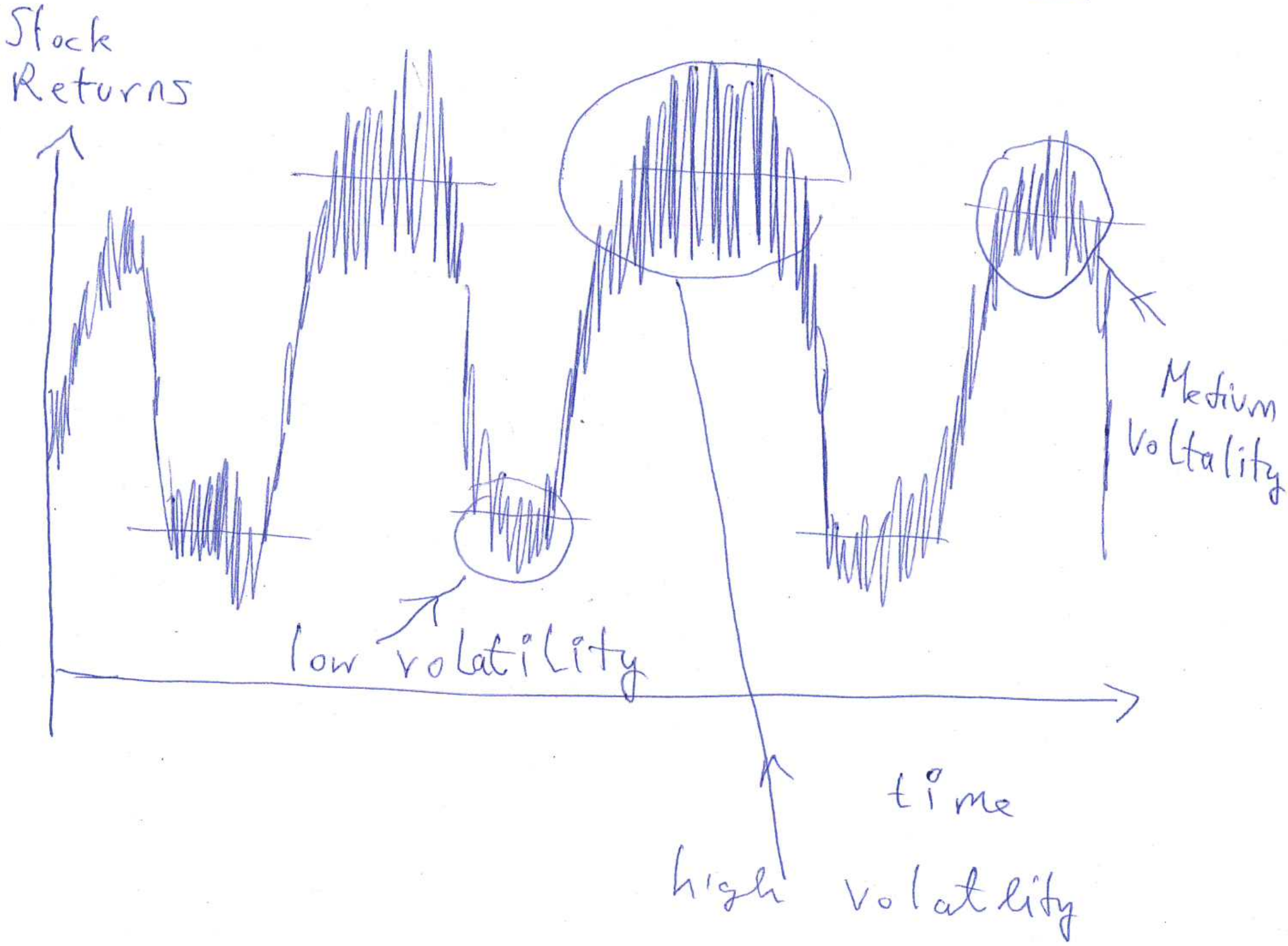
$$\Rightarrow (x+1)^a = \infty$$

$$\Rightarrow x+1 = \infty$$

$$\Rightarrow x = \infty$$

$$\Rightarrow w(F) = \infty$$

Models for Stock Returns



$\Rightarrow$ volatility is itself a random variable

$\Rightarrow$ Let X = Stock Returns

Suppose $X | \sigma^2 \sim N(\mu, \sigma^2)$

Let $g(\cdot)$ denote the PDF of σ^2

Then the ~~uncot~~ uncontitional PDF of X is

$$f_X(x) = \int_0^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}_{\text{PDF of } N(\mu,\sigma^2)} \underbrace{g(\sigma^2)}_{\text{PDF of } \sigma^2} \, d\sigma$$

Total Prob Law

X = stock returns (observable)

σ^2 = volatility (unobservable)

eg

$$\sigma^2 \sim \text{Uni}[a, b] \quad , \quad \begin{matrix} a > 0 \\ b > 0 \\ b > a \end{matrix}$$

$$f_X(x) = \int_0^\infty \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \frac{1}{b-a}\, d\sigma$$

$$= \frac{1}{\sqrt{2\pi}\,(b-a)} \int_a^b \frac{1}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, d\sigma$$

Set

$$y = \frac{(x-\mu)^2}{2\sigma^2}$$

$$\sigma = \frac{|x-\mu|}{\sqrt{2y}}$$

$$\frac{d\sigma}{dy} = -\frac{|x-\mu|}{2\sqrt{2}\, y^{3/2}}$$

$$= \frac{1}{\sqrt{2\pi}(b-a)} \int \frac{\sqrt{2y}}{|x-\mu|} e^{-y} (-1) \frac{|x-\mu|}{2\sqrt{2}\, y^{3/2}} \, dy$$

$$= \frac{1}{2\sqrt{2\pi}\,(b-a)} \int_{\frac{(x-\mu)^2}{2b^2}}^{\frac{(x-\mu)^2}{2a^2}} \cancel{\frac{1}{\sqrt{y}}} \frac{1}{y} e^{-y} \, dy$$

$$= \frac{1}{2\sqrt{2\pi}\,(b-a)} \left[\int_0^{\frac{(x-\mu)^2}{2a^2}} \frac{1}{y} e^{-y} \, dy - \int_0^{\frac{(x-\mu)^2}{2b^2}} \frac{1}{y} e^{-y} \, dy \right]$$

$$\boxed{f_X(x) = \frac{1}{2\sqrt{2\pi}\,(b-a)} \left[\gamma\left(0, \frac{(x-\mu)^2}{2a^2}\right) - \gamma\left(0, \frac{(x-\mu)^2}{2b^2}\right) \right]},$$

where

$$\boxed{\gamma(a, x) = \int_0^x t^{a-1} e^{-t} \, dt}$$

incomplete gamma function

eg σ^2 has the PDF $\frac{2}{\sigma^3} e^{-\frac{1}{\sigma^2}}$

Fréchet

$$f_X(x) = \int_0^\infty \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(X-\mu)^2}{2\sigma^2}} \cdot \frac{2}{\sigma^3} e^{-\frac{1}{\sigma^2}} \, d\sigma$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sigma^4} e^{-\frac{1}{\sigma^2}\left[\frac{(X-\mu)^2}{2}+1\right]} \, d\sigma$$

Set

$$y = \frac{1}{\sigma^2}\left[\frac{(X-\mu)^2}{2}+1\right]$$

$$\sigma = \frac{1}{\sqrt{y}} \sqrt{\frac{(X-\mu)^2}{2}+1}$$

$$\frac{d\sigma}{dy} = -\frac{1}{2y^{3/2}} \sqrt{\frac{(X-\mu)^2}{2}+1}$$

$$f_X(x) = \frac{2}{\sqrt{2\pi}} \int \frac{y^2}{\left[\frac{(x-\mu)^2}{2} + 1\right]^2} e^{-y}$$

$$\cdot (-1) \frac{1}{2y^{3/2}\sqrt{\frac{(x-\mu)^2}{2} + 1}}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{(x-\mu)^2}{2} + 1\right]^{-3/2} \boxed{\int_0^\infty y^{1/2} e^{-y}\, dy}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{(x-\mu)^2}{2} + 1\right]^{-3/2} \Gamma\left(\frac{3}{2}\right)$$

$$\left[\ \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}\ \right]$$

$$\boxed{f_X(x) = \frac{1}{2\sqrt{2}} \left[\frac{(x-\mu)^2}{2} + 1\right]^{-3/2}}$$

ETT

Suppose $X_1, X_2, \ldots, X_n$ IID with CDF F. Let $M_n = \max(X_1, \ldots, X_n)$. If there exists $a_n > 0$, $b_n \in \mathbb{R}$ and a non-degenerate CDF G such that

$$P\left(\frac{M_n - b_n}{a_n} < x\right) \to G(x)$$

as $n \to \infty$ then G must be of the same type as

$$\text{I}: \quad \Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R}$$

$$\text{II}: \quad \Phi_\alpha(x) = e^{-x^{-\alpha}}, \quad x > 0, \ \alpha > 0$$

$$\underline{\text{III}}: \quad \Psi_\alpha(x) = e^{-(-x)^\alpha}, \quad x < 0, \ \alpha < 0$$

ETT talks about the extremes of 1 variable.

Sometimes the interest is in the extremes of more than 1 variable.

eg (Oil Price, Gold Price)

(Rainfall, Wind Speed)

$P(X \leq x, Y \leq y) = F(x, y)$ Joint CDF

$F_Y(y) = F(\infty, y)$ Marginal CDF of Y

$F_X(x) = F(x, \infty)$ Marginal CDF of X

$f_Y(y) = \dfrac{dF_Y(y)}{dy}$ Marginal PDF of Y

$f_X(x) = \dfrac{dF_X(x)}{dx}$ Marginal PDF of X

Suppose $(X_1, Y_1), (X_2, Y_2),$

$\ldots, (X_n, Y_n)$ are IID with

joint CDF $F(\cdot, \cdot)$.

Define

$$M_{nX} = \max(X_1, X_2, \ldots, X_n)$$

$$M_{nY} = \max(Y_1, Y_2, \ldots, Y_n)$$

The bivariate extreme value is

$$(M_{nX}, M_{nY})$$

$$\Pr(M_{nX} \leq a_n x + b_n, M_{nY} \leq c_n y + d_n)$$

$$= \Pr(\max(X_1, \ldots, X_n) \leq a_n x + b_n,$$

$$\max(Y_1, \ldots, Y_n) \leq c_n y + d_n)$$

indep

$$= \Pr(X_1 \leq a_n x + b_n, Y_1 \leq c_n y + d_n)$$

$$\cdots \Pr(X_n \leq a_n x + b_n, Y_n \leq c_n y + d_n)$$

$$= F(a_n x + b_n, c_n y + d_n)$$

$$\cdots F(a_n x + b_n, c_n y + d_n)$$

$$= F^n(a_n x + b_n, c_n y + d_n)$$

$$\Rightarrow \Pr\left(\frac{M_n X - b_n}{a_n} \leq x, \frac{M_n Y - d_n}{c_n} \leq y\right)$$

$$= F^n(a_n x + b_n, c_n y + d_n)$$

$$\Rightarrow \lim_{n\to\infty} \Pr\left(\frac{M_n X - b_n}{a_n} \leq x, \frac{M_n Y - d_n}{c_n} \leq y\right)$$

$$= \lim_{n\to\infty} F^n(a_n x + b_n, c_n y + d_n)$$

If the limit is a non-degenerate CDF $G(\cdot,\cdot)$ then its form can take an uncountable forms.

Suppose a non-degenerate limit G exists. How do we determine a_n, b_n, c_n & d_n ?

$$\Rightarrow \lim_{n\to\infty} F^n(a_n x + b_n, c_n y + d_n) = G(x, y) \quad -(*)$$

Set $x = \infty \Rightarrow$

$$\lim_{n\to\infty} F^n(\infty, c_n y + d_n) = G(\infty, y)$$

$$\Rightarrow \lim_{n\to\infty} F_Y^n(c_n y + d_n) = G(\infty, y)$$

[where F_Y is the marginal CDF of F]

$\Rightarrow c_n = \gamma(F_Y^{-1}(1-\frac{1}{n}))$, $d_n = F_Y^{-1}(1-\frac{1}{n})$ if $G(\infty,y)$ is Gumbel

$c_n = F_Y^{-1}(1-\frac{1}{n})$ & $d_n = 0$ if $G(\infty, y)$ is Fréchet

$c_n = w(F_Y) - F_Y^{-1}(1-\frac{1}{n})$ & $d_n = w(F_Y)$ if $G(\infty,y)$ is Weibull

Set $y = \infty$ into $(*)$

$$\lim_{n\to\infty} F^n(a_n x + b_n, \infty) = G(x,\infty)$$

$$\Rightarrow \lim_{n\to\infty} F_X^n(a_n x + b_n) = G(x,\infty)$$

[where F_X is the marginal CDF of F]

By the ETT,

$a_n = \gamma(F_X^{-1}(1-\frac{1}{n}))$, $b_n = F_X^{-1}(1-\frac{1}{n})$

if $G(x,\infty)$ is Gumbel

$a_n = F_X^{-1}(1-\frac{1}{n})$ & $b_n = 0$

if $G(x,\infty)$ is Fréchet

$a_n = w(F_X) - F_X^{-1}(1-\frac{1}{n})$ & $b_n = w(F_X)$

if $G(x,\infty)$ is Weibull.

eg

$$F(x, y) = 1 - [1+x]^{-1} - [1+y]^{-1} + [1+x+y]^{-1}$$

$$F_Y(y) = F(\infty, y) = 1 - [1+y]^{-1}, \quad w(F_Y) = \infty$$

$$F_X(x) = F(x, \infty) = 1 - [1+x]^{-1}, \quad w(F_X) = \infty$$

$$\lim_{t \to \infty} \frac{1 - F_Y(ty)}{1 - F_Y(t)} = \lim_{t \to \infty} \frac{1 - (1 - [1+ty]^{-1})}{1 - (1 - [1+t]^{-1})}$$

$$= \lim_{t \to \infty} \frac{[1+ty]^{-1}}{[1+t]^{-1}} = y^{-1}$$

$\Rightarrow$ F_Y belongs to the Fréchet domain

$$\Rightarrow \quad c_n = F_Y^{-1}\left(1 - \frac{1}{n}\right), \quad d_n = 0$$

$$F_Y(y) = 1 - [1+y]^{-1}$$

$$F_Y^{-1}(y) = \frac{1}{1-y} - 1 = \frac{y}{1-y}$$

$$\Rightarrow \quad c_n = \frac{1 - \frac{1}{n}}{1 - (1 - \frac{1}{n})} = n - 1, \quad d_n = 0$$

Similarly, it can be shown

$$a_n = n-1, \quad b_n = 0$$

$$\lim_{n\to\infty} F^n(a_n x + b_n, c_n y + d_n)$$

$$= \lim_{n\to\infty} F^n((n-1)x, (n-1)y)$$

$$= \lim_{n\to\infty} \Big\{ 1 - [1+(n-1)x]^{-1} - [1-(n-1)y]^{-1} + [1+(n-1)x+(n-1)y]^{-1} \Big\}^n$$

$$= \lim_{n\to\infty} \Big\{ 1 - [(n-1)x]^{-1} - [(n-1)y]^{-1} + [(n-1)x + (n-1)y]^{-1} \Big\}^n$$

$$= \lim_{n\to\infty} \Big\{ 1 - \frac{1}{(n-1)x} - \frac{1}{(n-1)y} + \frac{1}{(n-1)(x+y)} \Big\}^n$$

$$= \lim_{n\to\infty} \left\{ 1 - \frac{n}{n-1} \cdot \frac{1}{nx} - \frac{n}{n-1} \cdot \frac{1}{ny} + \frac{n}{n-1} \cdot \frac{1}{n(x+y)} \right\}^n$$

$\frac{n}{n-1} \to 1$

$$= \lim_{n\to\infty} \left\{ 1 - \frac{1}{nx} - \frac{1}{ny} + \frac{1}{n(x+y)} \right\}^n$$

$$= \lim_{n\to\infty} \left\{ 1 + \frac{1}{n}\left(-\frac{1}{x} - \frac{1}{y} + \frac{1}{x+y} \right) \right\}^n$$

$$= e^{-\frac{1}{x} - \frac{1}{y} + \frac{1}{x+y}} = G(x,y)$$

$$\left[\left(1 + \frac{Z}{n}\right)^n \longrightarrow e^Z \text{ as } n \to \infty \right]$$

eg

$$F(x,y) = xy\left[1+\theta(1-x)(1-y)\right], \quad \begin{matrix} 0<x\leq 1 \\ 0<y<1 \end{matrix}$$

Q: $F^n(a_n x + b_n, c_n y + d_n) \to G(x,y)$ as $n \to \infty$?

i) $F(x,1) = x \cdot 1 \cdot [1+\theta\cdot(1-x)(1-1)] = x = F_X(x)$

$F(1,y) = 1 \cdot y \cdot [1+\theta\cdot(1-1)\cdot(1-y)] = y = F_Y(y)$

$w(F_X) = 1$, $w(F_Y) = 1$

$F_X^{-1}(x) = x$, $F_Y^{-1}(y) = y$

ii)

$$\lim_{t \downarrow 0} \frac{1-F_X(1-tx)}{1-F_X(1-t)} = \lim_{t\downarrow 0} \frac{1-(1-tx)}{1-(1-t)} = x$$

$a_n = 1 - F_X^{-1}(1-\frac{1}{n}) = 1-(1-\frac{1}{n}) = \frac{1}{n}$

$b_n = 1$

Similarly, $c_n = \frac{1}{n}$ & $d_n = 1$

(iii) $\lim\limits_{n\to\infty} F^n(a_n x + b_n, c_n y + d_n)$

$$= \lim_{n\to\infty} F^n\left(\frac{x}{n}+1, \frac{y}{n}+1\right)$$

$$= \lim_{n\to\infty} \left(\frac{x}{n}+1\right)^n \left(\frac{y}{n}+1\right)^n \cdot \left[1+\theta\left(-\frac{x}{n}\right)\left(-\frac{y}{n}\right)\right]^n$$

$$= \lim_{n\to\infty} \boxed{\left(\frac{x}{n}+1\right)^n} \cdot \left(\frac{y}{n}+1\right)^n \cdot \left(1+\frac{\theta x y}{n^2}\right)^n$$

$$= \lim_{n\to\infty} e^x \, e^y \, e^{\frac{\theta x y}{n}} \to 1$$

$$= e^{x+y} = G(x, y).$$

$$F^n(a_n x + b_n, c_n y + d_n) \xrightarrow[n \to \infty]{} G(x, y)$$

Possible forms for G are uncountably infinite.

If $G(x, \infty)$ & $G(\infty, y)$ are both Gumbel distributed then we can write

$$G(x,y) = e^{-\int_0^1 \min\left[f_1(s)e^{-x},\, f_2(s)e^{-y}\right] ds}$$

where $f_1(s)$, $f_2(s)$ are non-negative Lebesgue integrable functions with $\int_0^1 f_i(s)\,ds = 1$.

If $G(x,\infty)$ & $G(\infty,y)$ are both Gumbel distributed then we can write

$$G(x,y) = e^{-(e^{-x}+e^{-y})\,k(y-x)}$$

where $k(\cdot)$ satisfies

$$\lim_{t \to \pm\infty} k(t) = 1$$

$$\frac{d}{dt}\left\{(1+e^{-t})\,k(t)\right\} \leq 0$$

$$\frac{d}{dt}\left\{(1+e^{t})\,k(t)\right\} \geq 0$$

$$(1+e^{-t})\,k''(t) + (1-e^{-t})\,k'(t) \geq 0.$$

If $G(x,\alpha)$ & $G(\alpha,y)$ are Frechet distrib~~ut~~ed then we can write

$$G(x,y) = e^{-\left(\frac{1}{x}+\frac{1}{y}\right) A\left(\frac{x}{x+y}\right)}$$

where $A : [0,1] \longrightarrow \mathbb{R}^+$ satisfies

$$A(0) = A(1) = 1$$

$$\max(w, 1-w) \leq A(w) \leq 1 \quad \forall w$$

$A(\cdot)$ is convex

The form for $G(x, y)$ is not known when $G(x, \infty)$ and $G(\infty, y)$ are Weibull distributed.

This is an open problem.

If $G(x, \infty) = 1 - e^{-x}$

and $G(\infty, y) = 1 - e^{-y}$ then

we can write

$$\overline{G}(x, y) = e^{-(x+y)\, A\left(\frac{y}{x+y}\right)}$$

where $A : [0, 1] \to \mathbb{R}^{+}$ satisfies

$A(0) = A(1) = 1$

$\max(w, 1-w) \leq A(w) \leq 1 \quad \forall w$

$A(\cdot)$ is convex.

Sheet 9 Q1

X = Stock returns

$X \mid \lambda \sim Exp(\lambda)$

$\lambda \sim Exp(a)$

$$f_X(x) = \int_0^{\infty} \underline{f_{X|\lambda}(x|\lambda)} \; \underline{g(\lambda)} \, d\lambda$$

$$= \int_0^{\infty} \lambda e^{-\lambda x} \; a e^{-a\lambda} \, d\lambda$$

$$= a \int_0^{\infty} \lambda e^{-\lambda(x+a)} \, d\lambda$$

$$\boxed{f_X(x) = \frac{a}{(x+a)^2}}$$

Suppose $X_1, X_2, \ldots, X_n$ IID

$$L(a) = \prod_{i=1}^{n} \left[\frac{a}{(X_i + a)^2} \right] = a^n \prod_{i=1}^{n} (X_i + a)^{-2}$$

$$\log L(a) = n \log a - 2 \sum_{i=1}^{n} \log (X_i + a)$$

$$\frac{d \log L(a)}{da} = \frac{n}{a} - 2 \sum_{i=1}^{n} \frac{1}{X_i + a} .$$

The MLE of a is the root of

$$\frac{n}{a} = 2 \sum_{i=1}^{n} \frac{1}{X_i + a} .$$

Sheet 9, Q2

$$X \mid \lambda \sim \text{Exp}(\lambda)$$

$$\lambda \sim \text{Uni}[a, b]$$

$$f_X(x) = \int_a^b \lambda e^{-\lambda x} \frac{1}{b-a} \, d\lambda$$

$$= \frac{1}{b-a} \int_a^b \lambda e^{-\lambda x} \, d\lambda$$

$$= \frac{(Xa+1)e^{-Xa} - (Xb+1)e^{-Xb}}{X^2(b-a)} = \frac{(Xa+1)e^{-Xa} - (Xb+1)e^{-Xb}}{X^2(b-a)}$$

Suppose $X_1, \ldots, X_n$.

$$L(a,b) = \prod_{i=1}^{n} \frac{e^{-X_i a} - e^{-X_i b}}{b-a}$$

$$= (b-a)^{-n} \prod_{i=1}^{n} \left(e^{-X_i a} - e^{-X_i b}\right)$$

$$\log L(a,b) = -n \log(b-a) + \sum_{i=1}^{n} \log\left(e^{-X_i a} - e^{-X_i b}\right)$$

Suppose $X_1, \ldots, X_n$ are IID data.

$$L(a,b) = \prod_{i=1}^{n} \frac{(X_i a+1)e^{-X_i a} - (X_i b+1)e^{-X_i b}}{X_i^2 (b-a)}$$

$$\log L(a,b) = -n \log(b-a) - 2\sum_{i=1}^{n} \log X_i$$

$$+ \sum_{i=1}^{n} \log\left[(X_i a+1)e^{-X_i a} - (X_i b+1)e^{-X_i b}\right]$$

The MLEs of a & b are the simultaneous solns of

$$\frac{\partial \log L}{\partial a} = 0 \quad \& \quad \frac{\partial \log L}{\partial b} = 0$$

Sheet 9, Q3

$$X \mid \lambda \sim \text{Exp}(\lambda)$$

$$g(\lambda) = a\lambda^{a-1}, \quad 0 < \lambda < 1$$

$$f_X(x) = \int_0^1 \lambda e^{-\lambda x} \cdot a\lambda^{a-1} \, d\lambda$$

$$= a \int_0^1 \lambda^a e^{-\lambda x} \, d\lambda$$

$$\left[y = \lambda x, \quad \lambda = \frac{y}{x}, \quad d\lambda = \frac{dy}{x} \right]$$

$$= a \int \left(\frac{y}{x}\right)^a e^{-y} \frac{dy}{x}$$

$$= \frac{a}{x^{a+1}} \boxed{\int_0^x y^a e^{-y} \, dy}$$

$$= \frac{a}{x^{a+1}} \gamma(a+1, x)$$

"Incomplete gamma fun"

Suppose $X_1, \ldots, X_n$ IID data.

$$L(a) = \prod_{i=1}^{n} \left[\frac{a}{X_i^{a+1}} \gamma(a+1, x_i) \right]$$

$$= a^n \prod_{i=1}^{n} \left[X_i^{-a-1} \gamma(a+1, x_i) \right]$$

$$\log L(a) = n \log a - (a+1) \sum_{i=1}^{n} \log x_i$$

$$+ \sum_{i=1}^{n} \log \gamma(a+1, x_i)$$

The MLE of a is the root of

$$\frac{\partial \log L(a)}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \frac{1}{\gamma(a+1, x_i)} \frac{\partial \gamma(a+1, x_i)}{\partial a}$$

$$= 0$$

Summary of Last Fri's Class

- Characterization of the bivariate extreme value CDF $G(x, y)$ when $G(x, \infty)$ & $G(\infty, y)$ are Gumbel distributed
- Characterization of $G(x, y)$ when $G(x, \infty)$ & $G(\infty, y)$ are Fréchet distributed
- Characterization of $G(x, y)$ when $G(x, \infty) = 1 - e^{-x}$ & $G(\infty, y) = 1 - e^{-y}$.

Sheet 12, Q3 a

$$\bar{G}(x,y) = e^{-(x^a+y^a)^{\frac{1}{a}}}, \quad a > 1 \quad \cancel{0 < a < 1}$$

$$= e^{-(x+y)\cdot\left[\left(\frac{x}{x+y}\right)^a + \left(\frac{y}{x+y}\right)^a\right]^{\frac{1}{a}}}$$

$$\left[\left(\frac{x}{x+y}\right)^a + \left(\frac{y}{x+y}\right)^a\right]^{\frac{1}{a}} = A\left(\frac{y}{x+y}\right)$$

$$\Rightarrow \quad A(w) = \left[w^a + (1-w)^a\right]^{1/a}$$

(i) $A(0) = \left[0^a + (1-0)^a\right]^{\frac{1}{a}} = 1$ ✓

$A(1) = \left[1^a + (1-1)^a\right]^{\frac{1}{a}} = 1$ ✓

(ii) $A(w) \leq 1$

$$\Leftrightarrow \quad \left[w^a + (1-w)^a\right]^{1/a} \leq 1$$

$$\Leftrightarrow \quad w^a + (1-w)^a \leq 1 \quad ✓$$

$$w^a \leq w^1 = w \quad \text{— (1)}$$

$$(1-w)^a \leq (1-w)^1 = 1-w \quad \text{— (2)}$$

$$(1)+(2) \Rightarrow \quad w^a + (1-w)^a \leq w + 1 - w = 1 \quad ✓$$

$$A(w) \geq \max(w, 1-w)$$

$$\Leftrightarrow \quad A(w) \geq w \;\&\; A(w) \geq 1-w$$

$$\Leftrightarrow \quad \left[w^a + (1-w)^a\right]^{\frac{1}{a}} \geq w \;\&\; \left[w^a + (1-w)^a\right]^{\frac{1}{a}} \geq 1-w$$

$$\Leftrightarrow \quad w^a + (1-w)^a \geq w^a \checkmark \;\&\; w^a + (1-w)^a \geq (1-w)^a \checkmark$$

(iii) $$A(w) = \left[w^a + (1-w)^a\right]^{\frac{1}{a}}$$

$$A'(w) = \frac{1}{a}\left[w^a + (1-w)^a\right]^{\frac{1}{a}-1} \cdot \left(a w^{a-1} - a(1-w)^{a-1}\right)$$

$$= \left[w^a + (1-w)^a\right]^{\frac{1}{a}-1}\left(w^{a-1} - (1-w)^{a-1}\right)$$

$$A''(w) = \left(\frac{1}{a}-1\right)\left[w^a + (1-w)^a\right]^{\frac{1}{a}-2} \cdot a\left[w^{a-1} - (1-w)^{a-1}\right]^2$$

$$+ (a-1)\left[w^a + (1-w)^a\right]^{\frac{1}{a}-1} \cdot \left[w^{a-2} + (1-w)^{a-2}\right]$$

Show that $A''(w) \geq 0 \;\forall w$

Examples

eg 1. Show that $\bar{G}(x, y) = e^{-x-y}$ is a bivariate extreme value distn with unit exponential marginals.

$$\bar{G}(x, y) = e^{-x-y}$$

$$= e^{-(x+y)\cdot 1}$$

$\Rightarrow \quad A(w) \equiv 1 \quad \forall \ w$ in $(*)$

i) $A(0) = 1$ ✓, $A(1) = 1$ ✓

ii) $A(w) \equiv 1 \leq 1$ ✓

$A(w) \equiv 1 \geq \max(w, 1-w)$ ✓

iii) $A(w) \equiv 1$

$f(\cdot)$ is convex if and only if $f''(x) \geq 0 \ \forall \ x$

$A'(w) = 0$

$A''(w) = 0 \geq 0 \Rightarrow A(\cdot)$ is convex ✓

eg 2

$$\bar{G}(x,y) = e^{-\max(x,y)}$$

Show that this is a BEVD function

$$\bar{G}(x,y) = e^{-(x+y)\boxed{\max\left(\frac{x}{x+y}, \frac{y}{x+y}\right)}}$$

$$= A\left(\frac{x}{x+y}\right)$$

$$\Rightarrow A(w) = \max(w, 1-w)$$

(i) $A(0) = \max(0, 1-0) = 1$ ✓

$A(1) = \max(1, 1-1) = 1$ ✓

(ii) $A(w) = \max(w, 1-w) \leq 1$ ✓

$A(w) = \max(w, 1-w) \geq \max(w, 1-w)$ ✓

(iii) $A(w) = \max(w, 1-w) = \begin{cases} w & \text{if } w \geq 1/2 \\ 1-w & \text{if } w < 1/2 \end{cases}$

$$A'(w) = \begin{cases} 1 & \text{if } w \geq 1/2 \\ -1 & \text{if } w < 1/2 \end{cases}$$

$$A''(w) = \begin{cases} 0 & \text{if } w \geq 1/2 \\ 0 & \text{if } w < 1/2 \end{cases}$$

$$= 0 \geq 0$$

$\Rightarrow A(\cdot)$ is convex ✓

eg 3

$$\bar{G}(x,y) = e^{-\frac{\theta y^2}{x+y} + \theta y - x - y}, \quad 0<\theta<1$$

Show that this a BEVD function.

$$\bar{G}(x,y) = e^{-(x+y)\left[\frac{\theta y^2}{(x+y)^2} - \frac{\theta y}{x+y} + 1\right]}$$

$\parallel A\left(\frac{y}{x+y}\right)$

$\Rightarrow \quad A(w) = \theta w^2 - \theta w + 1$

(i) $\quad A(0) = \theta \cdot 0^2 - \theta \cdot 0 + 1 = 1 \quad \checkmark$

$\quad A(1) = \theta \cdot 1^2 - \theta \cdot 1 + 1 = 1 \quad \checkmark$

(ii) $\quad A(w) \leq 1$

$\Leftrightarrow \quad \theta w^2 - \theta w + 1 \leq 1$

$\Leftrightarrow \quad \theta w^2 - \theta w \leq 0$

$\Leftrightarrow \quad \theta w(w-1) \leq 0 \quad \checkmark$

$A(w) \geq \max(w, 1-w)$

$\Leftrightarrow \quad A(w) \geq w \quad \& \quad A(w) \geq 1-w$

$\Leftrightarrow \quad \theta w^2 - \theta w + 1 \geq w \quad \& \quad \theta w^2 - \theta w + 1 \geq 1-w$

$\Leftrightarrow \quad \theta w(w-1) + 1 - w \geq 0 \quad \& \quad \theta w^2 - \theta w + w \geq 0$

$\Leftrightarrow \quad (1-w)(1-\theta w) \geq 0 \quad \& \quad w(\theta w - \theta + 1) \geq 0$

$\Leftrightarrow \quad (1-w)(1-\theta w) \geq 0 \quad \& \quad w(\theta w + 1 - \theta) \geq 0 \quad \checkmark$

(iii)

$$A(w) = \theta w^2 - \theta w + 1$$

$$A'(w) = 2\theta w - \theta$$

$$A''(w) = 2\theta > 0$$

$\Rightarrow A(\cdot)$ is convex

$\Rightarrow \bar{G}(x, y)$ is a BEVD function.

eg 4

$$\bar{G}(x,y) = e^{-x-y+[x^{-\theta}+y^{-\theta}]^{-1/\theta}}, \quad \theta > 0$$

Show that this a BEVD function.

$$\bar{G}(x,y) = e^{-(x+y)\left\{1 - \frac{1}{x+y}[x^{-\theta}+y^{-\theta}]^{-\frac{1}{\theta}}\right\}}$$

$$= e^{-(x+y)\boxed{\left\{1 - \left[\left(\frac{x}{x+y}\right)^{-\theta} + \left(\frac{y}{x+y}\right)^{-\theta}\right]^{-\frac{1}{\theta}}\right\}}}$$

$$\shortparallel \quad A\left(\frac{y}{x+y}\right)$$

$$\Rightarrow \quad A(w) = 1 - [w^{-\theta} + (1-w)^{-\theta}]^{-\frac{1}{\theta}}$$

(i) $$A(0) = 1 - [0^{-\theta} + (1-0)^{-\theta}]^{-\frac{1}{\theta}}$$

$$= 1 - [\infty + 1]^{-\frac{1}{\theta}} = 1 \checkmark$$

$$A(1) = 1 - [1^{-\theta} + (1-1)^{-\theta}]^{-\frac{1}{\theta}}$$

$$= 1 - [1 + \infty]^{-\frac{1}{\theta}} = 1 \checkmark$$

(ii) $A(w) \leq 1$

$$\Leftrightarrow \quad 1-\left[w^{-\theta}+(1-w)^{-\theta}\right]^{-\frac{1}{\theta}} \leq 1$$

$$\Leftrightarrow \quad -\left[w^{-\theta}+(1-w)^{-\theta}\right]^{-\frac{1}{\theta}} \leq 0 \qquad \checkmark$$

$$A(w) \geq \max(w, 1-w)$$

$$\Leftrightarrow \quad A(w) \geq w \quad \& \quad A(w) \geq 1-w$$

$$\Leftrightarrow \quad 1-\left[w^{-\theta}+(1-w)^{-\theta}\right]^{-\frac{1}{\theta}} \geq w$$

$$\& \quad 1-\left[w^{-\theta}+(1-w)^{-\theta}\right]^{-\frac{1}{\theta}} \geq 1-w$$

$$\Leftrightarrow \quad 1-w \geq \left[w^{-\theta}+(1-w)^{-\theta}\right]^{-\frac{1}{\theta}}$$

$$\& \quad w \geq \left[w^{-\theta}+(1-w)^{-\theta}\right]^{-\frac{1}{\theta}}$$

$$\Leftrightarrow \quad \left[(1-w)^{-\theta}\right]^{-\frac{1}{\theta}} \geq \left[w^{-\theta}+(1-w)^{-\theta}\right]^{-\frac{1}{\theta}} \qquad \checkmark$$

$$\& \quad \left[w^{-\theta}\right]^{-\frac{1}{\theta}} \geq \left[w^{-\theta}+(1-w)^{-\theta}\right]^{-\frac{1}{\theta}} \qquad \checkmark$$

(iii) $A(w) = 1 - \left[w^{-\theta} + (1-w)^{-\theta}\right]^{-\frac{1}{\theta}}$

$$A'(w) = (-1)\left(-\frac{1}{\theta}\right)\left[w^{-\theta} + (1-w)^{-\theta}\right]^{-\frac{1}{\theta}-1} \cdot \left[(-\theta) w^{-\theta-1} + \theta(1-w)^{-\theta-1}\right]$$

$$= \left[w^{-\theta} + (1-w)^{-\theta}\right]^{-\frac{1}{\theta}-1} \cdot \left[-w^{-\theta-1} + (1-w)^{-\theta-1}\right]$$

$$A''(w) = \left(-\frac{1}{\theta}-1\right)\left[w^{-\theta} + (1-w)^{-\theta}\right]^{-\frac{1}{\theta}-2} \cdot \left[-\theta w^{-\theta-1} + \theta(1-w)^{-\theta-1}\right] \cdot \left[-w^{-\theta-1} + (1-w)^{-\theta-1}\right]$$

$$+ \left[w^{-\theta} + (1-w)^{-\theta}\right]^{-\frac{1}{\theta}-1} \cdot \left[(\theta+1) w^{-\theta-2} + (\theta+1)(1-w)^{-\theta-2}\right]$$

Show that $A''(w) \geq 0 \quad \forall w$

Sheet 12, Q2

$$\bar{G}(x,y) = e^{\frac{\alpha x y}{x+y} - x - y} \quad , 0 < \alpha < 1$$

$$= e^{-(x+y)\cdot\left[-\frac{\alpha x y}{(x+y)^2} + 1\right]}$$

$$\left[-\frac{\alpha x y}{(x+y)^2} + 1\right] = A\left(\frac{y}{x+y}\right)$$

$$A(w) = 1 - \alpha w (1-w)$$

(i) $A(0) = 1 - \alpha \cdot 0 \cdot (1-0) = 1$ ✓

$A(1) = 1 - \alpha \cdot 1 \cdot (1-1) = 1$ ✓

(ii) $A(w) \leq 1$

$\Leftrightarrow 1 - \alpha w (1-w) \leq 1$

$\Leftrightarrow -\alpha w (1-w) \leq 0$ ✓

$A(w) \geq w$ & $A(w) \geq 1-w$

$\Leftrightarrow 1 - \alpha w (1-w) \geq w$ & $1 - \alpha w (1-w) \geq 1-w$

$\Leftrightarrow 1 - w - \alpha w (1-w) \geq 0$ & $w - \alpha w (1-w) \geq 0$

$\Leftrightarrow \underline{(1-w)(1-\alpha w) \geq 0}$ ✓ & $w \cdot [1 - \alpha(1-w)] \geq 0$ ✓

(iii) $A(w) = 1 - \alpha w(1-w)$

$A'(w) = \cancel{-\alpha}(1-2w)$

$A''(w) = 2\alpha > 0$

$\Rightarrow A(\cdot)$ is convex

$\Rightarrow \bar{G}(x, y)$ is a BEVD.

Underreported/Overreported Income

In the economic literature, the under reported income is commonly expressed by the multiplicative relationship $Z = XY$, where Y is a multiplicative error and X denotes the true income. It is known that if Y has the power function distribution then X is Pareto distributed if and only if Z is also, see Krishnaji (1970).

The over reported income is commonly expressed by the multiplicative relationship $Z = X/Y$, where X and Y are independent random variables with X denoting the true income and Y a multiplicative error taking values in the interval $(0, 1)$. It is known that if Y has the power function distribution then X is Pareto distributed if and only if Z is also, see Krishnaji (1970).

A Pareto random variable has cdf specified by $F(x) = 1 - (K/x)^a$ for $x > K$. A power function random variable has cdf specified by $F(x) = x^c$ for $0 < x < 1$.

References

[1] Krishnaji, N. (1970). Characterization of the Pareto distribution through a model of under-reported incomes. *Econometrica*, **38**, 251-255.

Income modelling

eg/ Person earns £30,012

Reported income £30,000
" Under reported income "

eg/ person earns £90,991

Reported income £91,000
" Over reported income "

Let X = True Income " Unobservable "

Y = Multiplicative error taking values in $(0,1)$

$Z = XY$ = Under reported income
" Observable "

$\frac{X}{Y}$ = Over reported income.

1) <u>Unter</u> <u>reported</u> <u>income</u>

If Y has the power function distribation the X is Parto distributed $\boxed{\text{if and only if}}$ $Z=XY$ is also.

<u>Proof</u>: $F_Y(y) = y^c, \quad 0 < y < 1$

$F_X(x) = 1 - \left(\frac{K}{x}\right)^a, \quad x > K$

(i) Assume X is Pareto distributed

$$F_Z(z) = P(Z < z) = P(XY < z)$$

$$= P\left(X < \frac{z}{Y}\right)$$

Total Prob Rule

$$= \int_0^1 P\left(X < \frac{z}{y}\right) f_Y(y)\,dy$$

$$= \int_0^1 F_X\left(\frac{z}{y}\right) f_Y(y)\,dy$$

$$= \int_0^1 \left[1 - \left(\frac{Ky}{z}\right)^a\right] c\, y^{c-1}\,dy$$

$$= \int_0^1 c\, y^{c-1}\,dy - \frac{cK^a}{z^a}\int_0^1 y^{a+c-1}\,dy$$

$$= c \cdot \left[\frac{y^c}{c} \right]_0^1 - \frac{c K^a}{Z^a} \left[\frac{y^{a+c}}{a+c} \right]_0^1$$

$$= c \cdot \left(\frac{1}{c} - 0 \right) - \frac{c K^a}{Z^a} \cdot \left(\frac{1}{a+c} - 0 \right)$$

$$= 1 - \frac{c K^a}{(a+c) Z^a}$$

$$= 1 - \frac{\left(c^{\frac{1}{a}} K / (a+c)^{\frac{1}{a}} \right)^a}{Z^a}$$

$$F_Z(z) = 1 - \frac{(K^*)^a}{Z^a},$$ Pareto CDF with parameters a & K^*.

$\Rightarrow$ Z is Pareto distributed

(ii) Z is Pareto distributed with CDF

$$F_Z(z) = 1 - \left(\frac{L}{z}\right)^b, \quad z > L$$

$$F_X(x) = P(X < x)$$

$$= P\left(\frac{Z}{Y} < x\right)$$

Total Prob Rule

$$= P(Z < xY)$$

$$= \int_0^{1} P(Z < xy)\, f_Y(y)\, dy$$

$$= \int_0^{1} F_Z(xy)\, f_Y(y)\, dy$$

$$= \int_0^1 \left[1 - \left(\frac{L}{xy}\right)^b\right] \cdot c \cdot y^{c-1}\, dy$$

$$= \int_0^1 c \cdot y^{c-1}\, dy - \frac{cL^b}{x^b} \cdot \int_0^1 y^{c-b-1}\, dy$$

$$= c \cdot \left[\frac{y^c}{c}\right]_0^1 - \frac{c \cdot L^b}{x^b} \left[\frac{y^{c-b}}{c-b}\right]_0^1$$

$$= 1 - \frac{c \cdot L^b}{(c-b)\, x^b} = 1 - \frac{\left(c^{\frac{1}{b}} L / (c-b)^{\frac{1}{b}}\right)^b}{x^b}$$

$$= 1 - \left(\frac{L^*}{x}\right)^b$$, a Pareto CDF $\Rightarrow$ X is Pareto distributed

2) Over reported income

If Y has the power function distribution the X is Pareto distributed if and only if $Z = X/Y$ is also.

Proof: (i) Assume that X is Pareto distributed with CDF $F_X(x) = 1 - \left(\frac{k}{x}\right)^a, \ x > k.$

$$F_Z(z) = P(Z < z) = P\left(\frac{X}{Y} < z\right)$$

$$= P(X < z \cdot Y)$$

Total Prob Rule

$$= \int_0^1 P(X < z \cdot y)\, f_Y(y)\, dy$$

$$= \int_0^1 F_X(z \cdot y)\, f_Y(y)\, dy$$

$$= \int_0^1 \left[1 - \left(\frac{k}{z \cdot y}\right)^a\right] c \cdot y^{c-1}\, dy$$

$$= c \cdot \int_0^1 y^{c-1}\, dy - \frac{c \cdot k^a}{z^a} \int_0^1 y^{c-a-1}\, dy$$

$$= c \cdot \left[\frac{y^c}{c}\right]_0^1 - \frac{c \cdot k^a}{z^a} \left[\frac{y^{c-a}}{c-a}\right]_0^1$$

$$= c \cdot \left(\frac{1}{c} - 0\right) - \frac{c \cdot k^a}{z^a}\left(\frac{1}{c-a} - 0\right)$$

$$= 1 - \frac{c\, k^a}{(c-a)\, z^a}$$

$$= 1 - \frac{\left(c^{\frac{1}{a}} k \,/\, (c-a)^{\frac{1}{a}}\right)^a}{z^a}$$

$$= 1 - \left(\frac{k^*}{z}\right)^a$$, a Pareto CDF

$\Rightarrow$ z is Pareto distributed

(ii) Z is Pareto distributed with CDF

$$F_Z(z) = 1 - \left(\frac{L}{z}\right)^b, \quad z > K$$

$$F_X(x) = P(X < x) = P(ZY < x)$$

Total Prob Rule

$$= P\left(Z < \frac{x}{Y}\right) = \int_0^1 P\left(Z < \frac{x}{y}\right) f_Y(y)\,dy$$

$$= \int_0^1 F_Z\left(\frac{x}{y}\right) f_Y(y)\,dy$$

$$= \int_0^1 \left[1 - \left(\frac{Ly}{x}\right)^b\right] \cdot c y^{c-1}\,dy$$

$$= c\int_0^1 y^{c-1}\,dy - \frac{c \cdot L^b}{x^b}\int_0^1 y^{b+c-1}\,dy$$

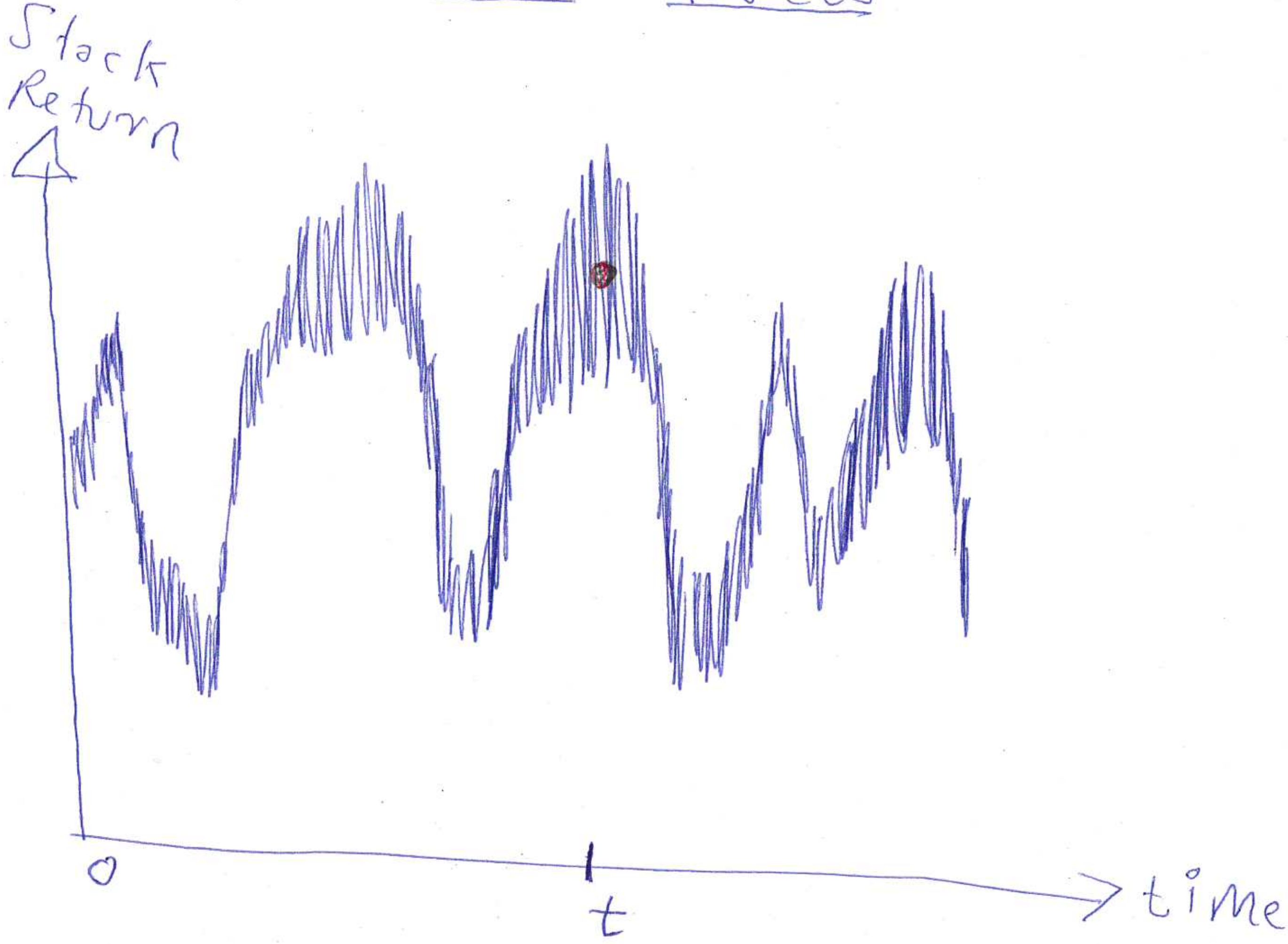
$$= 1 - \frac{c \cdot L^b}{(b+c)\,x^b}$$

$$= 1 - \frac{\left(c^{\frac{1}{b}} L \,/\, (b+c)^{\frac{1}{b}}\right)^b}{x^b}$$

$$= 1 - \left(\frac{L^*}{x}\right)^b, \text{ a Pareto CDF}$$

$\Rightarrow X$ is Pareto distributed

GARCH Models



Model non-IID data

Let X_t = Observation at time t

σ_t = Volatility at time t

X_t will depend on $X_{t-1}, X_{t-2}, \ldots$,

σ_t will depend on $\sigma_{t-1}, \sigma_{t-2}, \ldots$

1) <u>ARCH(q)</u> <u>model</u>

volatility at time t "unobserved"

$$X_t = \sigma_t Z_t$$

where $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \dots + \alpha_q X_{t-q}^2$

"observed"

Observed stock return at time t

"innovation" at time t "unobserved"

Commonly, Z_t is assumed to have

$Z_t \sim N(0,1)$ IID

or $Z_t \sim$ Student's $t\,(\nu)$

How to estimate the parameters $\alpha_0, \alpha_1, \dots, \alpha_q$?

$$X_t = \sigma_t Z_t$$

$$\Rightarrow \quad Z_t = \frac{X_t}{\sigma_t}$$

$$= \frac{X_t}{\sqrt{\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}}}$$

$$\sim N(0,1) \quad \text{IID}$$

$$\Rightarrow L(\alpha_0, \alpha_1, \ldots, \alpha_q)$$

$$= \prod_{t=1}^{n} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{X_t^2}{2(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q})}} \right]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} \sum_{i=1}^{n} \frac{X_t^2}{\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2}}$$

$$\Rightarrow \quad \log L = -\frac{n}{2} \log(2\pi)$$

$$- \frac{1}{2} \sum_{i=1}^{n} \frac{X_t^2}{\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2}$$

$$\frac{\partial \log L}{\partial \alpha_0} = \frac{1}{2} \sum_{i=1}^{n} \frac{X_t^2}{(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2)^2} = 0$$

$$\frac{\partial \log L}{\partial \alpha_1} = \frac{1}{2} \sum_{i=1}^{n} \frac{X_t^2 X_{t-1}^2}{(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2)^2} = 0$$

$$\vdots$$

$$\frac{\partial \log L}{\partial \alpha_q} = \frac{1}{2} \sum_{i=1}^{n} \frac{X_t^2 X_{t-q}^2}{(\alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_q X_{t-q}^2)^2} = 0$$

The MLEs of $\alpha_0, \alpha_1, \ldots, \alpha_q$ are the simultaneous solns of these eqns.

2) GARCH(p, q) model

$$X_t = \sigma_t Z_t$$

where

"observed"

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \dots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \dots + \beta_p \sigma_{t-p}^2$$

"not observed"

---

we cannot fit this model without an initial value for σ_0

$$E(X_t) = E(\sigma_t Z_t)$$
$$= E(\sigma_t)\, E(Z_t)$$
$$= E(\sigma_t) \cdot 0 = 0$$

$$E(X_t^2) = E(\sigma_t^2 Z_t^2)$$
$$= E(\sigma_t^2)\, E(Z_t^2) = E(\sigma_t^2) \cdot 1 = E(\sigma_t^2)$$

$$Var(X_t) = E(X_t^2) - (E(X_t))^2 = E(\sigma_t^2)$$

$$E(\sigma_t^2) = E\Big(\alpha_0 + \alpha_1 X_{t-1}^2 + \dots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \dots + \beta_p \sigma_{t-p}^2\Big)$$

$$\Rightarrow E(\sigma_t^2) = \alpha_0 + \alpha_1 E(X_{t-1}^2) + \dots + \alpha_q E(X_{t-q}^2) + \beta_1 \sigma_{t-1}^2 + \dots + \beta_p \sigma_{t-p}^2$$

$$\Rightarrow E(\sigma_t^2) = \alpha_0 + \alpha_1 E(\sigma_{t-1}^2) + \dots + \alpha_q E(\sigma_{t-q}^2) + \beta_1 E(\sigma_{t-1}^2) + \dots + \beta_p E(\sigma_{t-p}^2)$$

Assume stationarity of volatility.

$$\Rightarrow \sigma^2 = \alpha_0 + \alpha_1 \sigma^2 + \dots + \alpha_q \sigma^2 + \beta_1 \sigma^2 + \dots + \beta_p \sigma^2$$

$$\Rightarrow \boxed{\sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \dots - \alpha_q - \beta_1 - \dots - \beta_p}}$$

3) NGARCH model

$$X_t = \sigma_t Z_t$$

where

$$\sigma_t^2 = \omega + \alpha\left(X_{t-1} - \theta\,\sigma_{t-1}\right)^2 + \beta\,\sigma_{t-1}^2$$

observed

not observed

4) QGARCH model

$$X_t = \sigma_t Z_t$$

not observed

where

$$\sigma_t^2 = K + \alpha X_{t-1}^2 + \beta\,\sigma_{t-1}^2 + \phi X_{t-1}$$

observed

5) GJR-GARCH model

$$X_t = \sigma_t Z_t$$

where

$$\sigma_t^2 = K + \delta\sigma_{t-1}^2 + \alpha X_{t-1}^2 + \phi X_{t-1}^2 I_{t-1}$$

and $I_{t-1} = 0$ if $X_{t-1} \geq 0$,

$I_{t-1} = 1$ if $X_{t-1} < 0$

Q1, Sheet 13

$$X_t = \sigma_t Z_t$$

$$E(X_t) = E(\sigma_t)\,E(Z_t) = E(\sigma_t)\cdot 0 = 0$$

$$E(X_t^2) = E(\sigma_t^2 Z_t^2) = E(\sigma_t^2)\,E(Z_t^2)$$
$$= E(\sigma_t^2)\cdot 1$$
$$= E(\sigma_t^2)$$

$$\mathrm{Var}(X_t) = \underline{\underline{E(\sigma_t^2)}}.$$

Assuming stationarity,

$$E(\sigma_t^2) = E(\alpha_0 + \alpha_1 X_{t-1}^2 + \dots + \alpha_q X_{t-q}^2)$$

$$\Rightarrow E(\sigma_t^2) = \alpha_0 + \alpha_1 E(X_{t-1}^2) + \dots + \alpha_q E(X_{t-q}^2)$$

$$\Rightarrow E(\sigma_t^2) = \alpha_0 + \alpha_1 E(\sigma_{t-1}^2) + \dots + \alpha_q E(\sigma_{t-q}^2)$$

$$\Rightarrow \sigma^2 = \alpha_0 + \alpha_1 \sigma^2 + \dots + \alpha_q \sigma^2$$

$$\Rightarrow \sigma^2 = \frac{\alpha_0}{1 - \alpha_1 - \dots - \alpha_q}$$

$$\Rightarrow \boxed{\mathrm{Var}(X_t) = \frac{\alpha_0}{1 - \alpha_1 - \dots - \alpha_q}}$$

Q3, Sheet 13

$$X_t = \sigma_t Z_t$$

$$E(X_t) = E(\sigma_t) E(Z_t) = E(\sigma_t) \cdot 0 = 0$$

$$E(X_t^2) = E(\sigma_t^2) E(Z_t^2) = E(\sigma_t^2) \cdot 1 = E(\sigma_t^2)$$

$$Var(X_t) = E(\sigma_t^2)$$

Assuming stationarity,

$$E(\sigma_t^2) = E\left[\omega + \alpha(X_{t-1} - \theta\sigma_{t-1})^2 + \beta\sigma_{t-1}^2\right]$$

$$\Rightarrow E(\sigma_t^2) = \omega + \alpha E\left[X_{t-1}^2 - 2\theta X_{t-1}\sigma_{t-1} + \theta^2\sigma_{t-1}^2\right] + \beta E(\sigma_{t-1}^2)$$

$$\Rightarrow E(\sigma_t^2) = \omega + \alpha E(X_{t-1}^2) - 2\theta\alpha E(\sigma_{t-1}^2 Z_{t-1}) + \theta^2\alpha E(\sigma_{t-1}^2) + \beta E(\sigma_{t-1}^2)$$

$$\Rightarrow E(\sigma_t^2) = \omega + \alpha E(\sigma_{t-1}^2) - \cancel{2\theta\alpha \cdot E(\sigma_{t-1}^2) \cdot E(Z_{t-1})} + \theta^2\alpha E(\sigma_{t-1}^2) + \beta E(\sigma_{t-1}^2)$$

$$\Rightarrow \sigma^2 = \omega + \alpha\sigma^2 + \theta^2\alpha\sigma^2 + \beta\sigma^2$$

$$\Rightarrow \boxed{\sigma^2 = \frac{\omega}{1 - \alpha - \theta^2\alpha - \beta}} = Var(X_t)$$

Q4, Sheet 13

$$E(X_t) = E(\sigma_t Z_t) = E(\sigma_t)\,E(Z_t) = 0$$
$$E(X_t^2) = E(\sigma_t^2 Z_t^2) = E(\sigma_t^2)\cdot 1 = E(\sigma_t^2)$$
$$Var(X_t) = E(\sigma_t^2)$$

Assuming stationarity,

$$E(\sigma_t^2) = E\left[K + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2 + \phi X_{t-1}\right]$$
$$\Rightarrow E(\sigma_t^2) = K + \alpha E(X_{t-1}^2) + \beta E(\sigma_{t-1}^2) + \phi E(X_{t-1})$$
$$\Rightarrow E(\sigma_t^2) = K + \alpha \cdot E(\sigma_{t-1}^2) + \beta E(\sigma_{t-1}^2) + 0$$
$$\Rightarrow \sigma^2 = K + \alpha \sigma^2 + \beta \sigma^2$$
$$\Rightarrow \sigma^2 = \frac{K}{1-\alpha-\beta}$$

Q5, Sheet 13 $\quad E(X_t) = 0, \quad Var(X_t) = E(\sigma_t^2)$

Assuming stationarity,

$$E(\sigma_t^2) = E(k + \delta\sigma_{t-1}^2 + \alpha X_{t-1}^2 + \phi X_{t-1}^2 I_{t-1})$$

$$\Rightarrow E(\sigma_t^2) = k + \delta E(\sigma_{t-1}^2) + \alpha E(X_{t-1}^2) + \phi E(X_{t-1}^2 I_{t-1})$$

$$\Rightarrow E(\sigma_t^2) = k + \delta E(\sigma_{t-1}^2) + \alpha E(\sigma_{t-1}^2) + \phi \cdot E(X_{t-1}^2 \mid X_{t-1} < 0) \cdot P(X_{t-1} < 0)$$

$$\Rightarrow \sigma^2 = k + \delta\sigma^2 + \alpha\sigma^2 + \phi \cdot E(X^2 \mid X < 0) \cdot P(X < 0)$$

$$\Rightarrow \boxed{\sigma^2 = \frac{k + \phi E(X^2 \mid X < 0) \cdot P(X < 0)}{1 - \delta - \alpha}}$$

$$= Var(X_t)$$

MATH38181: Extreme values and financial risk
Semester 1
~~Solutions to~~ Problem sheet 11

1) For the standard generalized extreme value (GEV) distribution given by the cdf

$$F(x) = \exp\left\{-(1+\xi x)^{-1/\xi}\right\}$$

(where $1+\xi x > 0$), derive the following:

(a) the pdf,

(b) the nth moment,

(c) the mean,

(d) the variance.

2) For the standard GP ~~generalized extreme value (GEV)~~ distribution given by the cdf

$$F(x) = 1 - (1+\xi x)^{-1/\xi}$$

(where $x \geq 0$ if $\xi \geq 0$ and $0 < x < -1/\xi$ if $\xi < 0$), derive the following:

(a) the pdf,

(b) the nth moment,

(c) the mean,

(d) the variance.

Sheet 11, Q1

$$F(x) = e^{-(1+\xi x)^{-\frac{1}{\xi}}}$$

(a) $$f(x) = \frac{dF(x)}{dx}$$

$$= e^{-(1+\xi x)^{-\frac{1}{\xi}}} (-1)\left(-\frac{1}{\xi}\right)(1+\xi x)^{-\frac{1}{\xi}-1} \cdot \xi$$

$$= (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}}$$

(b) $$E(X^n) = \int x^n f(x)\, dx$$

$$= \int x^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

$$= \int \left(\frac{1+\xi x - 1}{\xi}\right)^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

$$= \frac{1}{\xi^n} \int (1+\xi x - 1)^n (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

$$= \frac{1}{\xi^n} \int \sum_{k=0}^{n} \binom{n}{k} (1+\xi x)^k (-1)^{n-k}$$
$$\cdot (1+\xi x)^{-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

$$= \frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int (1+\xi x)^{k-\frac{1}{\xi}-1} e^{-(1+\xi x)^{-\frac{1}{\xi}}} dx$$

Set $y = (1+\xi x)^{-\frac{1}{\xi}}$

$$y^{-\xi} = 1 + \xi x$$

$$x = \frac{y^{-\xi} - 1}{\xi}$$

$$\frac{dx}{dy} = \frac{(-\xi)\, y^{-\xi-1}}{\xi} = -y^{-\xi-1}$$

$$= \frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int (y^{-\xi})^{k-\frac{1}{\xi}-1} e^{-y}\, y^{-\xi-1} (-1)\, dy$$

$$= \frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int y^{-\xi k} e^{-y} (-1)\, dy$$

$\boxed{\xi > 0}$:

$$1 + \xi x = 0 \Rightarrow y = \infty$$
$$1 + \xi x = \infty \Rightarrow y = 0$$

$$= \frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_{\infty}^{0} y^{-\xi k} e^{-y} (-1)\, dy$$

$$= \frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \boxed{\int_{0}^{\infty} y^{-\xi k} e^{-y}\, dy}$$

$$\boxed{E(X^n) = \frac{1}{\xi^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}\, \Gamma(1-\xi k)} \quad \text{if } \xi > 0$$

(c) $E(X) = \frac{1}{\xi} \sum_{k=0}^{1} \binom{1}{k} (-1)^{1-k} \Gamma(1-\xi k)$

$= \frac{1}{\xi} \left[-\Gamma(1) + \Gamma(1-\xi) \right]$

$= \frac{1}{\xi} \left[-1 + \Gamma(1-\xi) \right]$ if $\xi > 0$

(d) $Var(X) = E(X^2) - (E(X))^2$

$E(X^2) = \frac{1}{\xi^2} \sum_{k=0}^{2} \binom{2}{k} (-1)^{2-k} \Gamma(1-\xi k)$

$= \frac{1}{\xi^2} \left[\Gamma(1) - 2\Gamma(1-\xi) + \Gamma(1-2\xi) \right]$

$= \frac{1}{\xi^2} \left[1 - 2\Gamma(1-\xi) + \Gamma(1-2\xi) \right]$

$Var(X) = \frac{1}{\xi^2} \left[1 - 2\Gamma(1-\xi) + \Gamma(1-2\xi) \right]$

$- \frac{1}{\xi^2} \left[-1 + \Gamma(1-\xi) \right]^2$

if $\xi > 0$

Sheet 11, Q1

$$F(x) = 1 - (1+\xi x)^{-\frac{1}{\xi}}$$

(a) $f(x) = (-1)\left(-\frac{1}{\xi}\right)(1+\xi x)^{-\frac{1}{\xi}-1}(\xi) = (1+\xi x)^{-\frac{1}{\xi}-1}$

(b) $$E(X^n) = \int x^n (1+\xi x)^{-\frac{1}{\xi}-1} dx$$

$$= \int \left(\frac{1+\xi x-1}{\xi}\right)^n (1+\xi x)^{-\frac{1}{\xi}-1} dx$$

$$= \frac{1}{\xi^n}\int \sum_{k=0}^{n}\binom{n}{k}(1+\xi x)^k(-1)^{n-k}(1+\xi x)^{-\frac{1}{\xi}-1} dx$$

$$= \frac{1}{\xi^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\int_0^\infty (1+\xi x)^{k-\frac{1}{\xi}-1} dx$$

$\boxed{\xi > 0}$:

$$= \frac{1}{\xi^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left[\frac{(1+\xi x)^{k-\frac{1}{\xi}}}{\left(k-\frac{1}{\xi}\right)\xi}\right]_0^\infty$$

$$= \frac{1}{\xi^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left[0 - \frac{1}{(k\xi-1)}\right]$$

$$\boxed{E(X^n) = \frac{1}{\xi^n}\sum_{k=0}^{n}\binom{n}{k}\cdot(-1)^{n-k}\frac{1}{1-k\xi}} \quad \text{if } \xi > 0$$

Revision

Sheet 12, Q1

$$\bar{F}(x,y) = e^{-\frac{\theta y^2}{x+y} + \theta y - x - y}$$

$$\left(\bar{F}(x,y) = P(X > x, Y > y) \right)$$

(b) $$\boxed{F(x,y) = 1 - \bar{F}(x,0) - \bar{F}(0,y) + \bar{F}(x,y)}$$

$$= 1 - e^{-x} - e^{-y} + e^{-\frac{\theta y^2}{x+y} + \theta y - x - y}$$

(c) $$F_{Y|X=x}(y|x) = \frac{\frac{\partial F(x,y)}{\partial x}}{\boxed{f_X(x)}}$$ PDF of X at x

$$\frac{\partial F(x,y)}{\partial x} = e^{-x} + e^{-\frac{\theta y^2}{x+y} + \theta y - x - y} \cdot \left[\frac{\theta y^2}{(x+y)^2} - 1 \right]$$

CDF of X at x

$$\boxed{F_X(x)} = F(x, \infty) = 1 - e^{-x}$$

$$\Rightarrow f_X(x) = \frac{d F_X(x)}{dx} = e^{-x}$$

$$F_{Y|X=x}(y|x) = 1 + e^{-\frac{\theta y^2}{x+y} + \theta y - y} \cdot \left[\frac{\theta y^2}{(x+y)^2} - 1 \right]$$

(d) Cond CDF X given $Y = y$ is

$$F_{X|Y=y}(x|y) = \frac{\frac{\partial F(x,y)}{\partial y}}{\boxed{f_Y(y)}}$$ ← PDF of Y at y

$$\frac{\partial F(x,y)}{\partial y} = e^{-y} + e^{-\frac{\theta y^2}{x+y} + \theta y - x - y} \cdot \left[-\frac{2\theta y}{x+y} + \frac{\theta y^2}{(x+y)^2} + \theta - 1 \right]$$

$$F_Y(y) = F(\infty, y) = 1 - e^{-y}$$

$$\Rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = e^{-y}$$

$$F_{X|Y=y}(x|y) = 1 + e^{-\frac{\theta y^2}{x+y} + \theta y - x} \cdot \left[-\frac{2\theta y}{x+y} + \frac{\theta y^2}{(x+y)^2} + \theta - 1 \right]$$

(e) $$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y} F(x,y)\right]$$
$$= \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x} F(x,y)\right]$$

(f) $$f_{Y|X=x}(y|x) = \frac{f(x,y)}{f_X(x)}$$

(g) $$f_{X|Y=y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$

$$p(x) = \frac{2^x e^{-2}}{x!}, \quad Po(2)$$

$$\lim_{k\to\infty} \frac{p(k)}{1-F(k-1)} = \lim_{k\to\infty} \frac{\frac{2^k \cancel{e^{-2}}}{k!}}{\sum_{j=k}^{\infty} \frac{2^j \cancel{e^{-2}}}{j!}}$$

$$\begin{aligned} & 1-F(k-1) \\ &= 1-P(X \le k-1) \\ &= P(X \ge k) \\ &= \sum_{j=k}^{\infty} P(X=j) \end{aligned}$$

$$= \lim_{k\to\infty} \frac{1}{\sum_{j=k}^{\infty} \frac{2^{j-k}\, k!}{j!}}$$

$$= \lim_{k\to\infty} \frac{1}{\sum_{j=k}^{\infty} \frac{2^{j-k}}{\underbrace{\boxed{(k+1)}}_{\ge k}\underbrace{\boxed{(k+2)}}_{\ge k}\cdots\underbrace{\boxed{(k+j-k)}}_{\ge k}}}$$

$$\ge \lim_{k\to\infty} \frac{1}{\sum_{j=k}^{\infty} \frac{2^{j-k}}{k^{j-k}}}$$

$$= \lim_{k\to\infty} \frac{1}{\sum_{j=k}^{\infty} \left(\frac{2}{k}\right)^{j-k}} \quad [m = j-k]$$

$$= \lim_{k\to\infty} \frac{1}{\sum_{m=0}^{\infty} \left(\frac{2}{k}\right)^{m}}$$

$$= \lim_{k\to\infty} \left(\frac{1}{\frac{1}{1-\frac{2}{k}}} \right) = 1$$

$$\Rightarrow \lim_{k\to\infty} \frac{p(k)}{1-F(k-1)} \ge 1 \Rightarrow$$ ETT cannot hold

$$F(x, y) = P(X < x, Y < y)$$

$$F_X(x) = F(x, \infty)$$

$$F_Y(y) = F(\infty, y)$$

$$\bar{F}(x, y) = P(X > x, Y > y)$$

$$F_X(x) = 1 - \bar{F}(x, 0)$$

$$F_Y(y) = 1 - \bar{F}(0, y)$$

$$F(x,y) = 1 - \bar{F}(x,0) - \bar{F}(0,y) + \bar{F}(x,y)$$

<u>Proof</u>:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Additive Law of Prob

$$\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$F(x,y,z) = 1 - \bar{F}(x,0,0) - \bar{F}(0,y,0) - \bar{F}(0,0,z) + \bar{F}(x,y,0) + \bar{F}(x,0,z) + F(0,y,z) - \bar{F}(x,y,z)$$

<u>Proof</u>: $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$ ← Add Law Prob

<u>EXAM 2013, Q4</u>

$$\bar{F}(x,y,z) = \left[1 + \frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right]^{-d}$$

(a) $M = \max(X, Y, Z)$

$$\boxed{F_M(m)} = P(M < m)$$

$$= P(\max(X,Y,Z) < m)$$

CDF of M $= P(X < m, Y < m, Z < m)$

$$= F(m, m, m)$$

$$= 1 + \bar{F}(0, m, m) + \bar{F}(m, 0, m) + \bar{F}(m, m, 0)$$

$$- \bar{F}(0, 0, m) - \bar{F}(0, m, 0) - \bar{F}(m, 0, 0)$$

$$- \bar{F}(m, m, m)$$

$$= 1 - \left[1 + \frac{m}{a}\right]^{-d} - \left[1 + \frac{m}{b}\right]^{-d} - \left[1 + \frac{m}{c}\right]^{-d}$$

$$+ \left[1 + \frac{m}{b} + \frac{m}{c}\right]^{-d} + \left[1 + \frac{m}{a} + \frac{m}{c}\right]^{-d} + \left[1 + \frac{m}{a} + \frac{m}{b}\right]^{-d}$$

$$- \left[1 + \frac{m}{a} + \frac{m}{b} + \frac{m}{c}\right]^{-d}$$

$$f_M(m) = \frac{d}{dm} F_M(m)$$

$$E(M^n) = \int_0^\infty m^n f_M(m)\, dm$$

$$= n \int_0^\infty m^{n-1} \left[1 - F_M(m)\right] dm$$

$$L = \min(X, Y, Z)$$

$$\begin{aligned}
F_L(l) &= P(L < l) \\
&= P(\min(X, Y, Z) < l) \\
&= 1 - P(\min(X, Y, Z) \geq l) \\
&= 1 - P(X \geq l, Y \geq l, Z \geq l) \\
&= 1 - \bar{F}(l, l, l) \\
&= 1 - \left[1 + \frac{l}{a} + \frac{l}{b} + \frac{l}{c}\right]^{-d}
\end{aligned}$$

$$f_L(l) = \frac{d}{dl} F_L(l)$$

$$\begin{aligned}
E(L^n) &= \int_0^\infty l^n f_L(l)\, dl \\
&= n \int_0^\infty l^{n-1} [1 - F_L(l)]\, dl
\end{aligned}$$

Revision

Suppose $\hat{\theta}$ estimates θ.

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

$$MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2$$

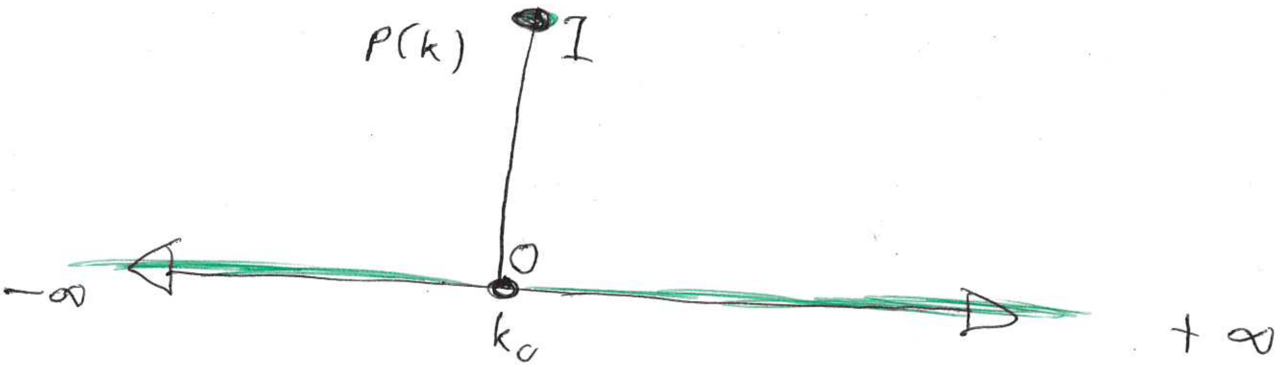
$\hat{\theta}$ is unbiased if $\text{Bias}(\hat{\theta}) = 0$

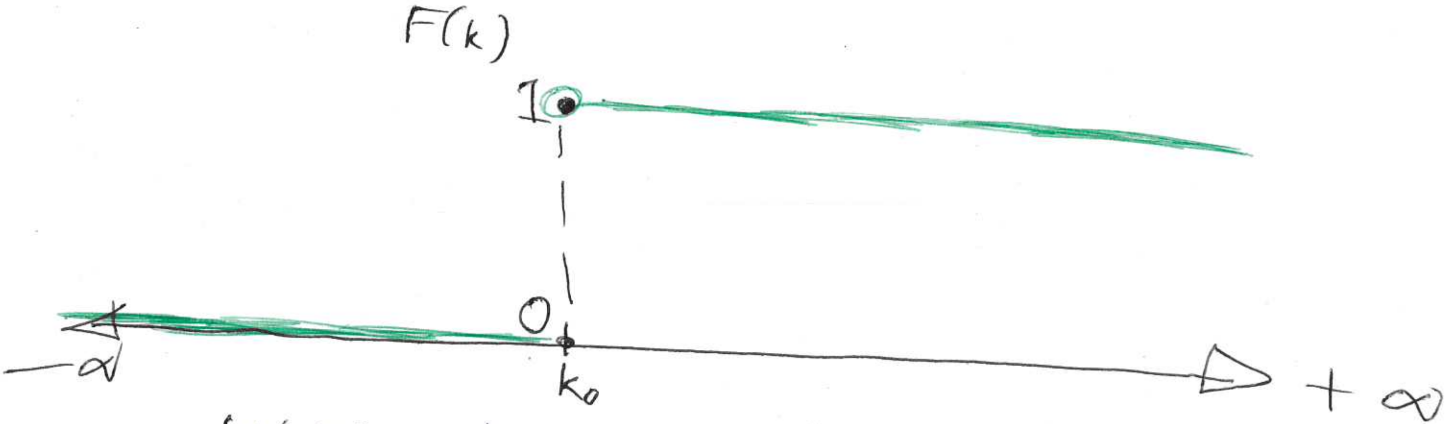
$MSE(\hat{\theta}) = Var(\hat{\theta})$ if $\hat{\theta}$ is unbiased

$\hat{\theta}$ is consistent if $\lim_{n \to \infty} MSE(\hat{\theta}) = 0$

$$p(k) = \begin{cases} 1 & \text{if} \quad k = k_0 \\ 0 & \text{if} \quad k \neq k_0 \end{cases}$$

$$\lim_{k \to \omega(F)} \frac{p(k)}{1 - F(k-1)}$$





$$\omega(F) = k_0$$

$$\lim_{k \to \omega(F)} \frac{p(k)}{1 - F(k-1)} = \frac{p(k_0)}{1 - F(k_0 - 1)} = \frac{1}{1 - 0} = 1$$

$\Rightarrow$ ETT cannot hold

$X \sim Exp(\lambda)$. Find $VaR_p(X)$ & $ES_p(X)$.

$$F(x) = 1 - e^{-\lambda x} = p$$

$$\Rightarrow \quad e^{-\lambda x} = 1 - p$$

$$\Rightarrow \quad -\lambda x = \log(1-p)$$

$$\Rightarrow \quad x = -\frac{1}{\lambda}\log(1-p)$$

$$\Rightarrow \quad VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_t(X)\,dt$$

$$= -\frac{1}{p\lambda}\int_0^p 1 \cdot \log(1-t)\,dt$$

Int by Parts

$$= -\frac{1}{p\lambda}\left\{\Big[t \cdot \log(1-t)\Big]_0^p - \int_0^p \frac{t}{(1-t)(-1)}\,dt\right\}$$

$$= -\frac{1}{p\lambda}\left\{p\log(1-p) - 0 + \int_0^p \frac{t}{1-t}\,dt\right\}$$

$$= -\frac{1}{p\lambda}\left\{p\log(1-p) + \int_0^p \frac{(t-1)+1}{1-t}\,dt\right\}$$

$$= -\frac{1}{p\lambda}\left\{p\log(1-p) + \int_0^p \left(-1 + \frac{1}{1-t}\right)dt\right\}$$

$$= -\frac{1}{p\lambda}\left\{p\log(1-p) + \Big[-t - \log(1-t)\Big]_0^p\right\}$$

$$= -\frac{1}{p\lambda}\left\{p\log(1-p) - p - \log(1-p) - 0\right\}$$

$$\left(1+\frac{x}{n}\right)^n \longrightarrow e^x \quad \text{as} \quad n \rightarrow \infty$$

$$e^x \approx 1+x \quad \text{if} \quad x \text{ is small}$$

$$(1+x)^a \approx 1+a \cdot x \quad \text{if} \quad x \text{ is small}$$

MGF

Defn $M_X(t) = E\left[e^{tX}\right]$

Prop

$$E(X) = M_X'(0)$$

$$E(X^2) = M_X''(0)$$

$$\vdots$$

$$E(X^n) = M_X^{(n)}(0)$$

If $X_1, X_2, \dots, X_n$ are ~~IID~~ indep

$$M_{X_1 + \cdots + X_n}(t) = \prod_{j=1}^{n} M_{X_j}(t)$$