Quantitative Finance

ISSN: 1469-7688 (Print) 1469-7696 (Online) Journal homepage: http://www.tandfonline.com/loi/rquf20

Financial Pareto ratios

Saralees Nadarajah & Samuel Kotz

To cite this article: Saralees Nadarajah & Samuel Kotz (2007) Financial Pareto ratios, Quantitative Finance, 7:3, 257-260, DOI: 10.1080/14697680601067604

To link to this article: http://dx.doi.org/10.1080/14697680601067604

Published online: 19 Jun 2007.

Quantitative Finance, Vol. 7, No. 3, June 2007, 257–260

Financial Pareto ratios

SARALEES NADARAJAH*† and SAMUEL KOTZ‡

†School of Mathematics, University of Manchester, Manchester M60 1QD, UK
‡Department of Engineering Management and Systems Engineering,
George Washington University, Washington, DC 20052, USA

(*Received 16 May 2006; in final form 16 October 2006*)

1. Introduction

Ratios of random variables arise most frequently in accounting. There are many financial indices that take the form of ratios. Some of the most commonly known examples are

(1) current ratio defined by Current assets (X)/Current liabilities (Y);
(2) sales margin defined by (Sales (X) $-$ Costs (Y))/Sales (X);
(3) changes in capital employed defined by (Closing capital (Y) $-$ Opening capital (X))/Opening capital (X);
(4) interest cover defined by (Earnings (X) + Interests paid (Y))/Earnings (X);
(5) liabilities ratio defined by Liabilities (X)/(Equity (Y) + Liabilities (X));
(6) financial leverage ratio defined by Liabilities (X)/(Total capital (Y) $-$ Liabilities (X)).

Each of the above ratios can be re-expressed as a function of $W = X/(X+Y)$, which takes values between 0 and 1. For example, the current ratio can be re-expressed as $W/(1-W)$ and the sales margin as $2 - 1/W$.

In this note, we consider the probability distribution of $W = X/(X+Y)$. We take X and Y to be independent Pareto random variables specified by the probability density functions (pdfs)

$$f_X(x) = \frac{ac^a}{(x+c)^{a+1}} \tag{1}$$

and

$$f_Y(y) = \frac{bd^b}{(y+d)^{b+1}}, \tag{2}$$

respectively, for $x > 0$, $y > 0$, $a > 0$, $b > 0$, $c > 0$ and $d > 0$. The Pareto distribution is chosen because it is the first and the most popular distribution used in accounting and finance. The recent book by Kleiber and Kotz (2003) describes it as the pillar of statistical 'size' distributions.

The exact expressions for the distribution of $W = X/(X+Y)$ are given in section 2. In section 3, we provide a tabulation of the associated percentage points as well as a computer program for generating similar tables. The analytical calculations involve the Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. The properties of the Gauss hypergeometric function can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2. Exact distribution of the ratio

Theorem 1 derives a representation for the pdf of $W = X/(X+Y)$ in terms of the Gauss hypergeometric function.

Theorem 1: *Suppose X and Y are independent Pareto random variables with pdfs* (1) *and* (2), *respectively. The pdf of* $W = X/(X+Y)$ *can be expressed as*:

$$f_W(w) = abcd^{-1} w^{-2} B(2, a+b) {}_2F_1 \times \left(2, b+1; a+b+2; 1 - \frac{c(1-w)}{dw}\right) \tag{3}$$

for $0 < w < 1$.

Proof: Transform $(R, W) = (X+Y, X/(X+Y))$. Under this transformation, the joint pdf of (R, W) is

$$f(r, w) = r f_X(rw) f_Y(r(1-w)) = \frac{abc^a d^b r}{(c+rw)^{a+1}(d+r(1-w))^{b+1}}. \tag{4}$$

*Corresponding author. Email: saralees.nadarajah@manchester.ac.uk

Quantitative Finance
ISSN 1469–7688 print/ISSN 1469–7696 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/14697680601067604

Using (4), the pdf of W can be expressed as

$$f_W(w) = \frac{abc^a d^b}{w^{a+1}(1-w)^{b+1}} \int_0^\infty \frac{r}{(r+c/w)^{a+1}(r+d/(1-w))^{b+1}} \mathrm{d}r. \quad (5)$$

The result of the theorem follows by using equation (2.2.6.24) in Prudnikov *et al.* (1986, volume 1) to calculate the integral in (6). □

Using special properties of the Gauss hypergeometric function, one can obtain several simpler expressions of (4). This is illustrated in the corollary below.

Corollary 1: *If $a \geq 1$ and $b \geq 1$ are integers then* (3) *can be reduced to the elementary form*

$$f_W(w) = \frac{abc(a+b+1)B(2,a+b)}{w(cw+\mathrm{d}w-c)} \times \left\{ G\left(b+1, a+b+2, 1-\frac{c(1-w)}{\mathrm{d}w}\right) - G\left(b, a+b+2, 1-\frac{c(1-w)}{\mathrm{d}w}\right) \right\}$$

for $0 < w < 1$, *where* $G(n,m,z)$ *is given by*

$$G(n,m,z) = \frac{(m-1)!}{(m-n-1)!z} \left\{ \sum_{k=1}^{m-n-1} \frac{(m-n-k-1)!}{(m-k-1)!} \times \left(\frac{z-1}{z}\right)^{k-1} - \frac{z}{(n-1)!}\left(\frac{z-1}{z}\right)^{m-n-1} \times \left[\sum_{k=1}^{n-1} \frac{z^{-k}}{n-k} + z^{-n}\log(1-z) \right] \right\}$$

for $m > n$ *and by*

$$G(n,m,z) = (m-1)(1-z)^{m-n-1} \sum_{k=0}^{n-m} \frac{(m-n)_k}{k!(m-k-1)} z^k$$

for $m \leq n$.

Figure 1 illustrates possible shapes of the pdf (4) for a range of values of a and b. The four curves in each plot correspond to selected values of b. The effect of the parameters is evident.

3. Percentiles

In this section, we provide tabulations of percentage points associated with the derived distribution of $W = X/(X+Y)$. These values are obtained by numerically solving the equation

$$abcd^{-1}B(2,a+b)\int_0^{w_q} w_2^{-2}F_1 \times \left(2, b+1; a+b+2; 1-\frac{c(1-w)}{\mathrm{d}w}\right)\mathrm{d}w = q.$$

Evidently, this involves computation of the Gauss hypergeometric function and routines for this are widely available. We used the function hypergeom $(\cdot)$ in the algebraic manipulation package, MAPLE. Table 1 provides the numerical values of w_q for $c=1$, $d=1$, $a = 1, 2, \ldots, 10$ and $b = a, a+1, \ldots, 10$.

These numbers can be used to determine the probabilities of the observed financial ratios mentioned in section 1. Similar tabulations could be easily derived for other values of q, a, b, c and d by using the hypergeom $(\cdot)$ function in MAPLE. A sample program is shown in the Appendix.

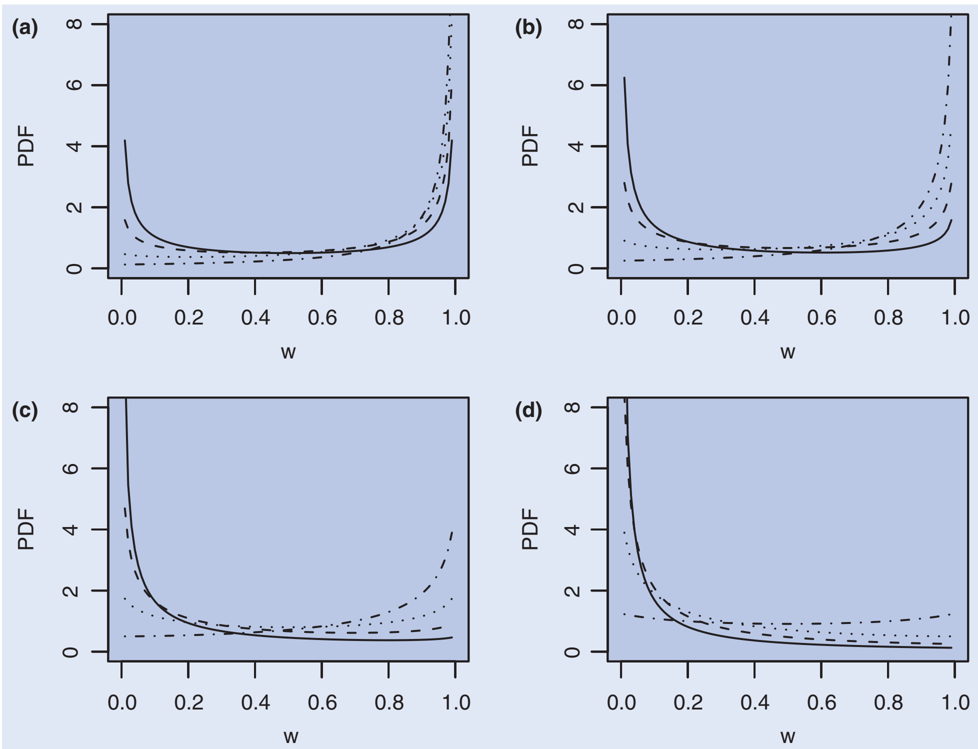

Figure 1. Plots of the pdf (3) for (*a*) $a = 0.5$; (*b*) $a = 1$; (*c*) $a = 2$; and (*d*) $a = 5$. The four curves in each plot correspond to $b = 0.5$ (solid curve), $b = 1$ (curve of dots), $b = 2$ (curve of dashes), and $b = 5$ (curve of dots and dashes).

Table 1. Percentage points w_q of $W = X/(X+Y)$.

a	b	$q = 0.01$	$q = 0.05$	$q = 0.1$	$q = 0.9$	$q = 0.95$	$q = 0.99$
1	1	0.002	0.015	0.040	0.960	0.985	0.998
1	2	0.011	0.060	0.131	0.982	0.993	0.999
1	3	0.020	0.105	0.212	0.988	0.995	0.999
1	4	0.030	0.147	0.281	0.991	0.997	1.000
1	5	0.039	0.185	0.339	0.993	0.997	1.000
1	6	0.049	0.220	0.388	0.994	0.998	1.000
1	7	0.058	0.252	0.430	0.995	0.998	1.000
1	8	0.067	0.282	0.467	0.996	0.998	1.000
1	9	0.076	0.309	0.500	0.996	0.999	1.000
1	10	0.084	0.335	0.529	0.997	0.999	1.000
2	2	0.005	0.030	0.066	0.934	0.970	0.995
2	3	0.010	0.054	0.113	0.956	0.980	0.996
2	4	0.015	0.077	0.156	0.967	0.985	0.997
2	5	0.020	0.099	0.195	0.974	0.988	0.998
2	6	0.025	0.121	0.231	0.978	0.990	0.998
2	7	0.030	0.141	0.264	0.981	0.992	0.998
2	8	0.034	0.160	0.294	0.984	0.993	0.999
2	9	0.039	0.179	0.322	0.985	0.993	0.999
2	10	0.044	0.197	0.348	0.987	0.994	0.999
3	3	0.007	0.036	0.077	0.923	0.964	0.993
3	4	0.010	0.052	0.108	0.942	0.973	0.995
3	5	0.013	0.068	0.137	0.953	0.978	0.996
3	6	0.017	0.083	0.164	0.961	0.982	0.997
3	7	0.020	0.098	0.190	0.966	0.984	0.997
3	8	0.023	0.112	0.215	0.971	0.986	0.997
3	9	0.026	0.126	0.237	0.974	0.988	0.998
3	10	0.030	0.139	0.259	0.976	0.989	0.998
4	4	0.008	0.039	0.082	0.918	0.961	0.992
4	5	0.010	0.052	0.105	0.933	0.968	0.994
4	6	0.013	0.063	0.128	0.944	0.974	0.995
4	7	0.015	0.075	0.149	0.952	0.977	0.996
4	8	0.017	0.086	0.169	0.958	0.980	0.996
4	9	0.020	0.097	0.188	0.962	0.982	0.997
4	10	0.022	0.108	0.206	0.966	0.984	0.997
5	5	0.008	0.042	0.086	0.914	0.958	0.992
5	6	0.010	0.051	0.104	0.928	0.965	0.993
5	7	0.012	0.061	0.122	0.938	0.970	0.994
5	8	0.014	0.070	0.139	0.945	0.974	0.995
5	9	0.016	0.079	0.156	0.951	0.977	0.996
5	10	0.018	0.088	0.171	0.956	0.979	0.996
6	6	0.008	0.043	0.088	0.912	0.957	0.992
6	7	0.010	0.051	0.103	0.924	0.963	0.993
6	8	0.012	0.059	0.118	0.933	0.968	0.994
6	9	0.013	0.067	0.133	0.940	0.971	0.994
6	10	0.015	0.074	0.147	0.946	0.974	0.995
7	7	0.009	0.044	0.090	0.910	0.956	0.991
7	8	0.010	0.051	0.103	0.921	0.961	0.992
7	9	0.011	0.058	0.116	0.929	0.966	0.993
7	10	0.013	0.064	0.128	0.936	0.969	0.994
8	8	0.009	0.045	0.091	0.909	0.955	0.991
8	9	0.010	0.051	0.102	0.918	0.960	0.992
8	10	0.011	0.057	0.114	0.926	0.964	0.993
9	9	0.009	0.045	0.092	0.908	0.955	0.991
9	10	0.010	0.051	0.102	0.917	0.959	0.992
10	10	0.009	0.046	0.093	0.907	0.954	0.991

References

Gradshteyn, I.S. and Ryzhik, I.M., *Table of Integrals, Series, and Products*, 6th ed., 2000 (Academic Press: San Diego).

Kleiber, C. and Kotz, S., *Statistical Size Distributions in Economics and Actuarial Sciences*, 2003 (John Wiley and Sons: New York).

Prudnikov, A.P., Brychkov, Y.A. and Marichev, O.I., *Integrals and Series*, Vols 1, 2 and 3, 1986 (Gordon and Breach Science Publishers: Amsterdam).

Appendix

The following program in MAPLE can be used to generate tables similar to that presented in section 3.

```
#this program generates percentiles of the product W=X/(X+Y)

ff:=a*b*(c/d)*Beta(2, a+b)*w**(-2):
ff:=ff*hypergeom([2,b+1],[a+b+2],1-c*(1−w)/(d*w)):
f:=int(ff,w=0..ww):

p1:=fsolve(f=0.6, ww=0..1):
p2:=fsolve(f=0.7, ww=0..1):
p3:=fsolve(f=0.8, ww=0..1):
p4:=fsolve(f=0.90, ww=0..1):
p5:=fsolve(f=0.95, ww=0..1):
p6:=fsolve(f=0.99, ww=0..1):
print(a, b, p1, p2, p3, p4, p5, p6);
```