MATH48181/68181: Extreme values and financial risk
Semester 1
Formulas to remember for the final exam in January 2022

The definition of a bivariate copula: $C(u_1, u_2)$ **is a bivariate copula if**

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

Definition of a bivariate extreme value distribution with unit exponential margins: a bivariate distribution with the joint survival function

$$\overline{G}(x, y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right]$$

for $x > 0$ **and** $y > 0$**, where** $A(\cdot)$ **is a real valued function on the unit interval satisfying: i)** $A(0) = A(1) = 1$**; ii)** $\max(w, 1-w) \leq A(w) \leq 1$ **for all** $w \in [0, 1]$**; and, iii)** $A(\cdot)$ **is convex.**

Extremal type theorem: Suppose $X_1, X_2, \ldots$ **are independent and identically distributed (iid) random variables with common cumulative distribution function** F**. Let** $M_n = \max\{X_1, \ldots, X_n\}$ **denote the maximum of the first** n **random variables and let** $w(F) = \sup\{x : F(x) < 1\}$ **denote the upper end point of** F**. If there are norming constants** $a_n > 0$**,** b_n **and a nondegenerate** G **such that the cdf of a normalized version of** M_n **converges to** G**, i.e.**

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \to G(x)$$

as $n \to \infty$ **then** G **must be of the same type as (cumulative distribution functions** G **and** G^* **are of the same type if** $G^*(x) = G(ax+b)$ **for some** $a > 0$**,** b **and all** x**) as one of**

the following three classes:

$$
\begin{aligned}
I &: \quad \Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \Re; \\
II &: \quad \Phi_\alpha(x) = \begin{cases} 0 & \textbf{if } x < 0\textbf{,} \\ \exp\left\{-x^{-\alpha}\right\} & \textbf{if } x \geq 0 \end{cases} \\
& \quad \textbf{for some } \alpha > 0; \\
III &: \quad \Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & \textbf{if } x < 0\textbf{,} \\ 1 & \textbf{if } x \geq 0 \end{cases} \\
& \quad \textbf{for some } \alpha > 0\textbf{.}
\end{aligned}
$$

Necessary and sufficient conditions for the three extreme value distributions:

$$
\begin{aligned}
I &: \quad \exists \gamma(t) > 0 \textbf{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x > 0, \\
II &: \quad w(F) = \infty \textbf{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: \quad w(F) < \infty \textbf{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}
$$

The corresponding formulas for a_n and b_n:

$$
\begin{aligned}
I &: \quad a_n = \gamma\left(F^{-1}\left(1 - n^{-1}\right)\right) \textbf{ and } b_n = F^{-1}\left(1 - n^{-1}\right), \\
II &: \quad a_n = F^{-1}\left(1 - n^{-1}\right) \textbf{ and } b_n = 0, \\
III &: \quad a_n = w(F) - F^{-1}\left(1 - n^{-1}\right) \textbf{ and } b_n = w(F),
\end{aligned}
$$

Necessary and sufficient conditions for $\left(M_n - b_n\right)/a_n$ to have a non-degenerate limiting distribution:

$$
\frac{\Pr(X = k)}{1 - F(k - 1)} \to 0
$$

as $k \to w(F)$ if F is discrete.

Definition of $\mathrm{VaR}_p(X)$ if X is an absolutely continuous random variable:

$$
\mathrm{VaR}_p(X) = F^{-1}(p).
$$

Definition of $\mathrm{ES}_p(X)$ if X is an absolutely continuous random variable:

$$
\mathrm{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.
$$

L'Hôpital's rule:

$$
\lim_{x \to c} \frac{f_1(x)}{f_2(x)} = \lim_{x \to c} \frac{f_1^{'}(x)}{f_2^{'}(x)}
$$

if $\lim_{x \to c} f_1(x) = \lim_{x \to c} f_2(x) = 0$ or $\pm\infty$.

The fact that: $(1-x)^a \approx 1 - ax$ for x close to zero.

The fact that: $\exp(-x) \approx 1 - x$ for x close to zero.

Joint survival function of $X_1, X_2, \ldots, X_k$ is defined by

$$\overline{F}\left(x_1, x_2, \ldots, x_k\right) = \Pr\left(X_1 > x_1, X_2 > x_2, \ldots, X_k > x_k\right).$$

Bivariate case: given the joint survival function $\overline{F}(x,y) = \Pr(X \geq x, Y \geq y)$ of two non-negative random variables, the marginal cdf of X, the marginal cdf of Y, the marginal pdf of X, the marginal pdf of Y, the joint cdf of (X,Y), the conditional cdf of Y given $X = x$ and the conditional cdf of X given $Y = y$ can be derived through

$$F_X(x) = 1 - \overline{F}\left(x, 0\right),$$

$$F_Y(y) = 1 - \overline{F}\left(0, y\right),$$

$$f_X(x) = \frac{dF_X(x)}{dx},$$

$$f_Y(y) = \frac{dF_Y(y)}{dy},$$

$$F_{X,Y}(x,y) = 1 - \overline{F}\left(x, 0\right) - \overline{F}\left(0, y\right) + \overline{F}\left(x, y\right),$$

$$F(y \mid x) = \frac{1}{f_X(x)} \frac{\partial F(x,y)}{\partial x}$$

and

$$F(x \mid y) = \frac{1}{f_Y(y)} \frac{\partial F(x,y)}{\partial y},$$

respectively.

The roots of $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.