MATH48181: Extreme values and financial risk
Semester 1
Formulas to remember for the final exam in January 2019

The definition of a bivariate copula: $C(u_1, u_2)$ **is a bivariate copula if**

$$C(u, 0) = 0,$$

$$C(0, u) = 0,$$

$$C(1, u) = u,$$

$$C(u, 1) = u,$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) \geq 0$$

and

$$\frac{\partial}{\partial u_2} C(u_1, u_2) \geq 0.$$

Definition of a bivariate extreme value distribution with unit exponential margins: a bivariate distribution with the joint survival function

$$\overline{G}(x, y) = \exp\left[-(x + y)A\left(\frac{y}{x + y}\right)\right]$$

for $x > 0$ **and** $y > 0$**, where** $A(\cdot)$ **is a real valued function on the unit interval satisfying: i)** $A(0) = A(1) = 1$**; ii)** $\max(w, 1 - w) \leq A(w) \leq 1$ **for all** $w \in [0, 1]$**; and, iii)** $A(\cdot)$ **is convex.**

Extremal type theorem: Suppose $X_1, X_2, \ldots$ **are independent and identically distributed (iid) random variables with common cumulative distribution function (cdf)** F**. Let** $M_n = \max\{X_1, \ldots, X_n\}$ **denote the maximum of the first** n **random variables and let** $w(F) = \sup\{x : F(x) < 1\}$ **denote the upper end point of** F**. If there are norming constants** $a_n > 0$, b_n **and a nondegenerate** G **such that the cdf of a normalized version of** M_n **converges to** G**, i.e.**

$$\Pr\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \to G(x)$$

as $n \to \infty$ **then** G **must be of the same type as (cdfs** G **and** G^* **are of the same type if** $G^*(x) = G(ax + b)$ **for some** $a > 0$, b **and all** x**) as one of the following three classes:**

$$I : \Lambda(x) = \exp\{-\exp(-x)\}, \qquad x \in \Re;$$

$$II : \Phi_\alpha(x) = \begin{cases} 0 & \textbf{if } x < 0, \\ \exp\{-x^{-\alpha}\} & \textbf{if } x \geq 0 \end{cases}$$

for some $\alpha > 0$;

$$III : \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & \textbf{if } x < 0, \\ 1 & \textbf{if } x \geq 0 \end{cases}$$

for some $\alpha > 0$.

Necessary and sufficient conditions for the three extreme value distributions:

$$
\begin{aligned}
I &: \exists \gamma(t) > 0 \textbf{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F\left(t + x\gamma(t)\right)}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II &: w(F) = \infty \textbf{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: w(F) < \infty \textbf{ and } \lim_{t \downarrow 0} \frac{1 - F\left(w(F) - tx\right)}{1 - F\left(w(F) - t\right)} = x^{\alpha}, \qquad x > 0.
\end{aligned}
$$

The corresponding formulas for a_n and b_n:

$$
\begin{aligned}
I &: a_n = \gamma\left(F^{-1}\left(1 - n^{-1}\right)\right) \textbf{ and } b_n = F^{-1}\left(1 - n^{-1}\right), \\
II &: a_n = F^{-1}\left(1 - n^{-1}\right) \textbf{ and } b_n = 0, \\
III &: a_n = w(F) - F^{-1}\left(1 - n^{-1}\right) \textbf{ and } b_n = w(F),
\end{aligned}
$$

Necessary and sufficient conditions for $\left(M_n - b_n\right)/a_n$ to have a non-degenerate limiting distribution:

$$
\frac{\Pr(X = k)}{1 - F(k-1)} \to 0
$$

as $k \to w(F)$ if F is discrete.

Definition of $\mathrm{VaR}_p(X)$ if X is an absolutely continuous random variable:

$$
\mathrm{VaR}_p(X) = F^{-1}(p).
$$

Definition of $\mathrm{ES}_p(X)$ if X is an absolutely continuous random variable:

$$
\mathrm{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.
$$

The definition of a moment generating function: $M_X(t) = E\left[\exp(tX)\right]$.

The probability mass and cumulative distribution functions of a Geometric(θ) random variable are:

$$
p(k) = \theta(1-\theta)^{k-1}
$$

and

$$
F(k) = 1 - (1-\theta)^k
$$

for $\theta > 0$ and $k = 1, 2, \ldots$.

Definition of beta function:

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

for $a > 0$ **and** $b > 0$**.**

Definition of incomplete beta function:

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$$

for $a > 0$**,** $b > 0$ **and** $0 < x < 1$**.**

Definition of incomplete beta function ratio:

$$I_x(a,b) = B_x(a,b)/B(a,b)$$

for $a > 0$**,** $b > 0$ **and** $0 < x < 1$**.**

L'Hôpital's rule:

$$\lim_{x \to c} \frac{f_1(x)}{f_2(x)} = \lim_{x \to c} \frac{f_1^{'}(x)}{f_2^{'}(x)}$$

if $\lim_{x \to c} f_1(x) = \lim_{x \to c} f_2(x) = 0$ **or** $\pm\infty$**.**

The fact that: $(1-x)^a \approx 1 - ax$ **for** x **close to zero.**

The fact that: $(1 + \frac{x}{n})^n \to e^x$ **as** $n \to \infty$**.**

Joint survival function of $X_1, X_2, \ldots, X_k$ **is defined by**

$$\overline{F}\left(x_1, x_2, \ldots, x_k\right) = \Pr\left(X_1 > x_1, X_2 > x_2, \ldots, X_k > x_k\right).$$

Bivariate case: given the joint survival function $\overline{F}(x,y) = \Pr(X \geq x, Y \geq y)$ **of two non-negative random variables, the marginal cdf of** X**, the marginal cdf of** Y**, the marginal pdf of** X**, the marginal pdf of** Y**, the joint cdf of** (X,Y)**, the conditional cdf of** Y **given** $X = x$ **and the conditional cdf of** X **given** $Y = y$ **can be derived through**

$$F_X(x) = 1 - \overline{F}\left(x, 0\right),$$

$$F_Y(y) = 1 - \overline{F}\left(0, y\right),$$

$$f_X(x) = \frac{dF_X(x)}{dx},$$

$$f_Y(y) = \frac{dF_Y(y)}{dy},$$

$$F_{X,Y}(x,y) = 1 - \overline{F}\left(x, 0\right) - \overline{F}\left(0, y\right) + \overline{F}\left(x, y\right),$$

$$F(y \mid x) = \frac{1}{f_X(x)} \frac{\partial F(x,y)}{\partial x}$$

and

$$F(x \mid y) = \frac{1}{f_Y(y)} \frac{\partial F(x,y)}{\partial y},$$

respectively.