MATH38181: Extreme values and financial risk
Semester 1
Formulas to remember for the final exam in January 2019

Extremal type theorem: Suppose $X_1, X_2, \ldots$ are independent and identically distributed (iid) random variables with common cumulative distribution function (cdf) F. Let $M_n = \max\{X_1, \ldots, X_n\}$ denote the maximum of the first n random variables and let $w(F) = \sup\{x : F(x) < 1\}$ denote the upper end point of F. If there are norming constants $a_n > 0$, b_n and a nondegenerate G such that the cdf of a normalized version of M_n converges to G, i.e.

$$\Pr\left(\frac{M_n - b_n}{a_n} \le x\right) = F^n\left(a_n x + b_n\right) \to G(x)$$

as $n \to \infty$ then G must be of the same type as (cdfs G and G^* are of the same type if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x) as one of the following three classes:

$$\begin{aligned}
I &: \quad \Lambda(x) = \exp\{-\exp(-x)\}, \qquad x \in \Re; \\
II &: \quad \Phi_\alpha(x) = \begin{cases} 0 & \textbf{if } x < 0, \\ \exp\{-x^{-\alpha}\} & \textbf{if } x \ge 0 \end{cases} \\
&\quad\ \textbf{for some } \alpha > 0; \\
III &: \quad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & \textbf{if } x < 0, \\ 1 & \textbf{if } x \ge 0 \end{cases} \\
&\quad\ \textbf{for some } \alpha > 0.
\end{aligned}$$

Necessary and sufficient conditions for the three extreme value distributions:

$$\begin{aligned}
I &: \quad \exists \gamma(t) > 0 \textbf{ s.t. } \lim_{t \uparrow w(F)} \frac{1 - F(t + x\gamma(t))}{1 - F(t)} = \exp(-x), \qquad x \in \Re, \\
II &: \quad w(F) = \infty \textbf{ and } \lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad x > 0, \\
III &: \quad w(F) < \infty \textbf{ and } \lim_{t \downarrow 0} \frac{1 - F(w(F) - tx)}{1 - F(w(F) - t)} = x^{\alpha}, \qquad x > 0.
\end{aligned}$$

Necessary and sufficient conditions for $(M_n - b_n)/a_n$ to have a non-degenerate limiting distribution:

$$\frac{\Pr(X = k)}{1 - F(k-1)} \to 0$$

as $k \to w(F)$ if F is discrete.

Definition of $\mathrm{VaR}_p(X)$ if X is an absolutely continuous random variable:

$$\mathrm{VaR}_p(X) = F^{-1}(p).$$

Definition of $\mathrm{ES}_p(X)$ if X is an absolutely continuous random variable:

$$\mathrm{ES}_p(X) = \frac{1}{p} \int_0^p F^{-1}(v) dv.$$

The definition of a moment generating function: $M_X(t) = E\left[\exp(tX)\right]$.

The probability mass and cumulative distribution functions of a Geometric(θ) random variable are:

$$p(k) = \theta(1-\theta)^{k-1}$$

and

$$F(k) = 1 - (1-\theta)^k$$

for $\theta > 0$ and $k = 1, 2, \ldots$.

Definition of beta function:

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

for $a > 0$ and $b > 0$.

Definition of incomplete beta function:

$$B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$$

for $a > 0$, $b > 0$ and $0 < x < 1$.

Definition of incomplete beta function ratio:

$$I_x(a, b) = B_x(a, b) / B(a, b)$$

for $a > 0$, $b > 0$ and $0 < x < 1$.

L'Hôpital's rule:

$$\lim_{x \to c} \frac{f_1(x)}{f_2(x)} = \lim_{x \to c} \frac{f_1^{'}(x)}{f_2^{'}(x)}$$

if $\lim_{x \to c} f_1(x) = \lim_{x \to c} f_2(x) = 0$ or $\pm\infty$.

The fact that: $(1-x)^a \approx 1 - ax$ for x close to zero.

Joint survival function of $X_1, X_2, \ldots, X_k$ is defined by

$$\overline{F}\left(x_1, x_2, \ldots, x_k\right) = \Pr\left(X_1 > x_1, X_2 > x_2, \ldots, X_k > x_k\right).$$