LECTURE

23 NOVEMBER

10:00-11:00AM

MATH4/68181

Copulas

A copula is a function $C : \underbrace{[0,1] \times [0,1]}_{\text{unit square}} \to [0,1]$

which describes the dependence between two variables, say X and Y.

Egs

1) X = gold price

   Y = silver "

2) X = gold price

   Y = diamond "

3) X = petrol price

   Y = car price

Let (X, Y) be a random vector with joint CDF $F(x, y)$. Then

$$F_X(x) = F(x, \infty)$$

$$F_Y(y) = F(\infty, y).$$

For every $F(x, y)$ there is a corresponding copula

$$F(x, y) = P(X \leq x, Y \leq y)$$

$$= P(F_X(X) \leq F_X(x), F_Y(Y) \leq F_Y(y))$$

probability integral transform

$$= P(U \leq F_X(x), V \leq F_Y(y))$$

$U \sim \text{Uni}[0, 1]$
$V \sim \text{Uni}[0, 1]$

$$\overset{0 \leq \quad \leq 1 \quad\quad 1 \leq U \leq 1}{= F_{U,V}(\boxed{F_X(x)}, F_Y(y))}$$

$$= C(F_X(x), F_Y(y))$$

$C : [0, 1] \times [0, 1] \to [0, 1]$

For every copula there is a corresponding $F(x, y)$

$$C(u, v) = P(U \leq u, V \leq v)$$

$$= P(F_X^{-1}(U) \leq F_X^{-1}(u), F_Y^{-1}(V) \leq F_Y^{-1}(v))$$

probability integral transform

$$= P(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(v))$$

$$= F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v))$$

Two extreme cases of copula

1) Complete independent copula

$$C(u, v) = u \cdot v$$

corresponds to

$$F_{X,Y}(x, y) = F_X(x)\, F_Y(y)$$

$\Longleftrightarrow$ X and Y are indep RVs

2) Complete dependent copula

$$C(u, v) = \min(u, v)$$

corresponds to

$$F_{X,Y}(x, y) = \min\left[F_X(x), F_Y(y)\right]$$

$$\begin{aligned}
&\quad \parallel \\
&P(X \leq x, Y \leq y) \\
&= P(X \leq x, X \leq y) \\
&= P(X \leq \min(x, y)) \\
&= F_X(\min(x, y)) \\
&= \min\left[F_X(x), F_X(y)\right]
\end{aligned}$$

Two most popular models for copulas

1) Normal (Gaussian) copula

$$C(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \times e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}} \, dy \, dx$$

where $-1 < \rho < 1$ (correlation coefficient) and $\Phi(\cdot)$ is the CDF of $N(0,1)$.

Gaussian copula is not a good model to describe dependence between financial variables.

Please read "The formula that killed Wall Street".

2) <u>t copula</u>

$$C(u,v) = \int_{-\infty}^{T_a^{-1}(u)} \int_{-\infty}^{T_a^{-1}(v)} \frac{\Gamma\left(\frac{a}{2}+1\right)}{\pi a \Gamma\left(\frac{a}{2}\right)\sqrt{1-\rho^2}}$$

$$\times \left[1 + \frac{x^2+y^2-2\rho xy}{a(1-\rho^2)}\right]^{-\frac{a+2}{2}} dy\, dx$$

where $-1 < \rho < 1$ (correlation coefficient)

a = degree of freedom

$T_a(\cdot)$ = CDF of the univariate Students't distribution with degree of freedom a.

t copula is a popular model to describe dependence between financial variablo.

Formal definition of a copula

A function $C : [0,1] \times [0,1] \rightarrow [0,1]$ is a copula if it satisfies the following:

i) $C(u, 0) = 0$

ii) $C(0, v) = 0$

iii) $C(u, 1) = u$

iv) $C(1, v) = v$

v) $\frac{\partial}{\partial u} C(u, v) \geqslant 0 \quad \forall\, u, v$

vi) $\frac{\partial}{\partial v} C(u, v) \geqslant 0 \quad \forall\, u, v$

Ex 1

$$C(u_1, u_2) = u_1 u_2$$

i) $C(u, 0) = u \cdot 0 = 0$ ✓

ii) $C(0, v) = 0 \cdot v = 0$ ✓

iii) $C(u, 1) = u \cdot 1 = u$ ✓

iv) $C(1, v) = 1 \cdot v = v$ ✓

v) $\frac{\partial}{\partial u} C(u, v) = v \geq 0$ ✓

vi) $\frac{\partial}{\partial v} C(u, v) = u \geq 0$ ✓

So, $C(u, v) = u \cdot v$ is a copula.

Ex 2

$$C(u_1, u_2) = u_1 u_2 e^{-\theta(\log u_1)(\log v_2)}$$

$$= u_1 v_2 u_2^{-\theta \log u_1} = u_1 u_2^{1-\theta \log u_1}$$

i) $C(u, 0) = \cancel{u \cdot 0 \cdot e} \quad u \cdot 0^{1-\theta \log u_1} = 0$

ii) $C(0, v) = 0 \cdot u_2^{\infty} = 0$

iii) $C(u, 1) = u \cdot 1 \cdot e^{-0} = u$

iv) $C(1, v) = 1 \cdot v \cdot e^{-0} = v$

v) $\frac{\partial}{\partial u_1} C(u_1, u_2) = u_2 e^{-\theta(\log u_1)(\log u_2)}$

$$+ u_1 v_2 e^{-\theta(\log u_1)(\log u_2)} \left(-\frac{\theta}{u_1}\right)$$

$$= u_2 e^{-\theta(\log u_1)(\log u_2)} [1 - \theta]$$