LECTURE

20 NOVEMBER

9:00-10:00AM

MATH3/4/68181

b) Non-parametric estimation methods

i) Historical method

Data : $x_1, x_2, \ldots, x_n$

Order the data as

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$$

Then

$$\widehat{ES}_p(X) = \frac{1}{[np]} \sum_{i=1}^{[np]} x_{(i)}$$

where $[x]$ denotes the largest integer less than or equal to x.

Ex $[4.4] = 4$, $[5.1] = 5$

Ex $-2 \quad 8 \quad 9 \quad -10 \quad 1$

$$\begin{array}{ccccc} -10 & -2 & 1 & 8 & 9 \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ x_{(1)} & x_{(2)} & x_{(3)} & x_{(4)} & x_{(5)} \end{array}$$

$$\widehat{ES}_{0.4}(X) = \frac{1}{2} \sum_{i=1}^{2} x_{(i)}$$

$$= \frac{1}{2}(-10-2)$$

$$= -6.$$

2) Bootstrap method

Data $x_1, x_2, \ldots, x_n$

(i) Compute the empirical CDF

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} I\{x_i \leqslant x\}$$

(ii) Simulate B samples each of size n from $\hat{F}$.

(iii) Compute $\widehat{ES}_p$ for each of the B samples by the historical method, resulting $\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \ldots, \widehat{ES}_p^{(B)}$.

(iv) Compute $\widehat{ES}_p$ as

$$\text{mean}\left(\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \ldots, \widehat{ES}_p^{(B)}\right)$$

or

$$\text{median}\left(\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \ldots, \widehat{ES}_p^{(B)}\right).$$

3) Jackknife method

Data $x_1, x_2, \ldots, x_n$

(i) Compute $\widehat{ES}_p$ by the historical method for $x_2, x_3, \ldots, x_n$, resulting in $\widehat{ES}_p^{(1)}$.

(ii) Compute $\widehat{ES}_p$ by the historical method for $x_1, x_3, \ldots, x_n$, resulting in $\widehat{ES}_p^{(2)}$

$\vdots$

(iii) Compute $\widehat{ES}_p$ by the historical method for $x_1, x_2, \ldots, x_{n-1}$, resulting in $\widehat{ES}_p^{(n)}$.

(iv) Compute $\widehat{ES}_p$ as

$$\text{Mean}\left(\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \ldots, \widehat{ES}_p^{(n)}\right)$$

or

$$\text{median}\left(\widehat{ES}_p^{(1)}, \widehat{ES}_p^{(2)}, \ldots, \widehat{ES}_p^{(n)}\right).$$

4) Kernel method

Data $x_1, x_2, \dots, x_n$

Order the data as $x_{(1)} \leqslant x_{(2)} \leqslant \dots \leqslant x_{(n)}$

The kernel estimator of $ES_p(X)$ is

$$\widehat{ES}_p(X) = \frac{1}{np} \sum_{i=1}^{n} x_i \, A_h\left(\hat{q}(p) - x_i\right)$$

where

$$h = \text{bandwidth}$$

$$\hat{q}(p) = \sum_{i=1}^{n} \left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(t-p)\, dt \right] x_{(i)}$$

$$K_h(u) = \frac{1}{h} K\left(\frac{u}{h}\right)$$

$$A_h(u) = A\left(\frac{u}{h}\right)$$

$$A(x) = \int_{-\infty}^{x} K(u)\, du$$

$K(\cdot)$ = kernel PDF usually chosen as the PDF of $N(0, 1)$

5) Richardson's method (based on Richardson (1911))

Richardson was a professor of applied math in Manchester. His picture is on the front door of G207, ground floor of ATB.

Suppose $x_1, x_2, \ldots, x_n$ are the data

(i) Compute the empirical CDF

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} I\{x_i \leqslant x\}$$

(ii) simulate $x_1, x_2, \ldots, x_N$ from $\hat{F}(\cdot)$

(iii) compute $\widehat{ES}_p$ by the historical method for the data simulated in (ii)

(iv) Repeat steps (ii) and (iii) 1000 times

(v) Let m_N = the average of the 1000 estimates of ES_p

(vi) Let $S_k = m_{N_k}, \quad k = 1, 2, \ldots, \ell+1$

for some ℓ and $N_1, N_2, \ldots, N_{\ell+1}$

(vii) Compute $\widehat{ES}_p$ as

$$\widehat{ES}_p = \frac{\begin{vmatrix} s_1 & s_2 & \cdots & s_{\ell+1} \\ 1 & \frac{1}{2} & \cdots & \frac{1}{\ell+1} \\ \vdots & & & \\ 1^\ell & \left(\frac{1}{2}\right)^\ell & \cdots & \left(\frac{1}{\ell+1}\right)^\ell \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \frac{1}{2} & \cdots & \frac{1}{\ell+1} \\ \vdots & \vdots & & \\ 1^\ell & \left(\frac{1}{2}\right)^\ell & \cdots & \left(\frac{1}{\ell+1}\right)^\ell \end{vmatrix}}$$

Ex

$$\ell = 1$$

$$N_1 = 100$$

$$N_2 = 1000$$

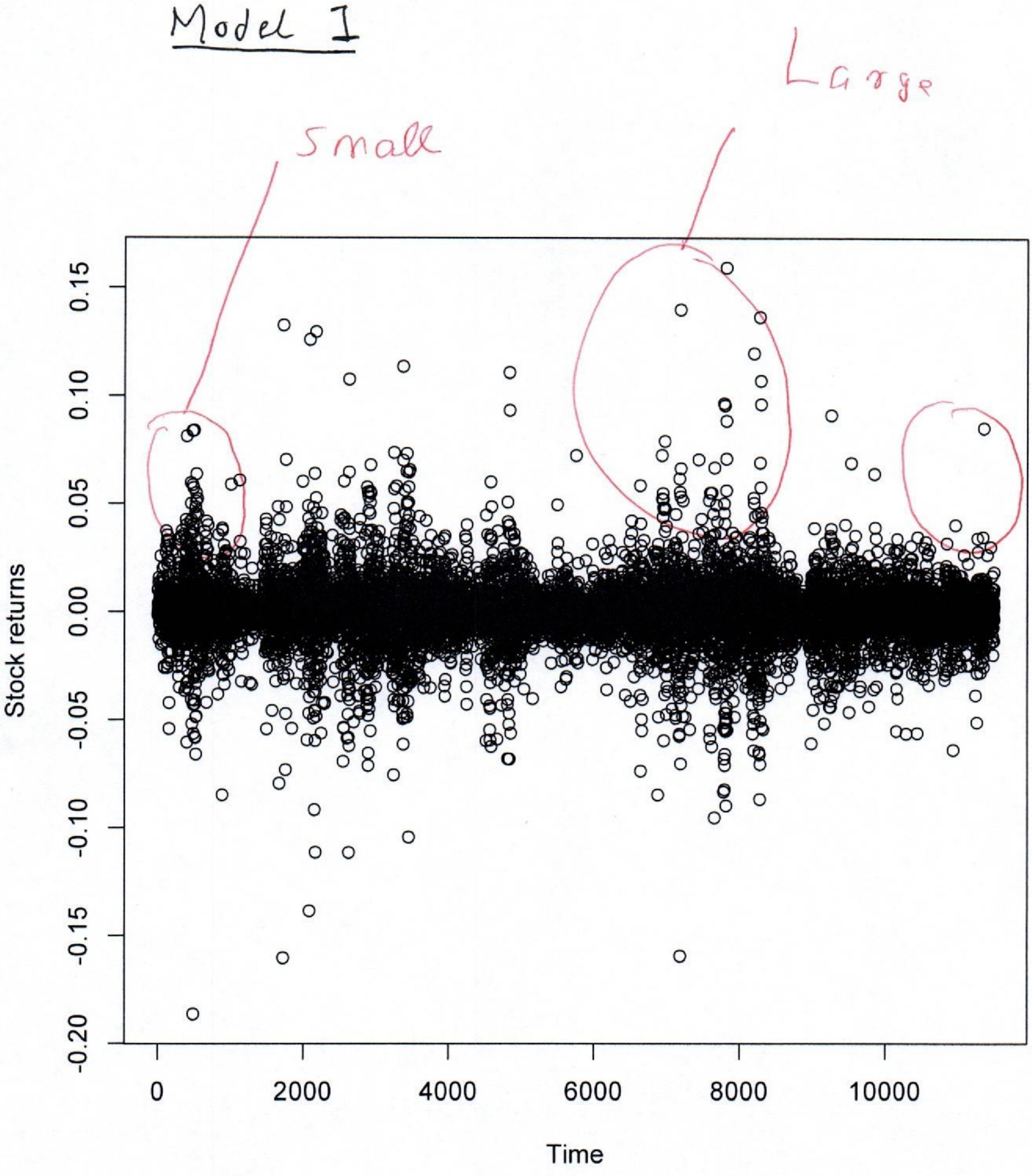
$$\widehat{ES}_p = \frac{\begin{vmatrix} m_{100} & m_{1000} \\ 1 & \frac{1}{2} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{vmatrix}}$$

$$= \frac{\frac{m_{100}}{2} - m_{1000}}{\frac{1}{2} - 1}$$

Models for Stock Returns

1) Model based on varying volatility
(Model 1)
2) Model based on additive random walk
(Model 2)
3) Model based on multiplicative random walk
(Model 3)

Model I



Small
Large
0.15
0.10
0.05
0.00
-0.05
-0.10
-0.15
-0.20
Stock returns
0
2000
4000
6000
8000
10000
Time

Model I

X = stock returns

V = variability
(volatility)

V is a random variable itself

V = unobservable

$X \mid V$ = observable

Hence, the PDF of X is

$$f_X(x) = \int_0^\infty \underbrace{f_{X|V}(x|v)}_{\substack{\text{conditional} \\ \text{PDF of } X|V}} \underbrace{f_V(v)}_{\text{PDF of } V} \, dv$$

The CDF of X is

$$F_X(x) = \int_0^\infty F_{X|V}(x|v) \, f_V(v) \, dv$$