EXAMPLE CLASS

20 NOVEMBER

16:00-17:00PM

MATH3/4/68181

Q1

$$L(\sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$

$$= \sigma^{-n} e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} e^{-\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -\frac{1}{\sigma}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}} - n\log\sigma$$

$$\frac{d\log L}{d\sigma} = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}} - \frac{n}{\sigma}$$

$$= 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}} + n\sigma$$

This must be solved numerically for the MLE $\hat{\sigma}$ of σ.

Q2

$$L(\sigma, \lambda) = \prod_{i=1}^{n} \left[\lambda \sigma^{+\lambda} x_i^{-\lambda-1} e^{-\left(\frac{\sigma}{x_i}\right)^\lambda} \right]$$

$$= \lambda^n \sigma^{n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{-\lambda-1} e^{-\sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^\lambda}$$

$$\log L(\sigma, \lambda) = n \log \lambda + n\lambda \log \sigma$$

$$- (\lambda+1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^\lambda$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^\lambda \frac{\lambda}{\sigma} = 0 \quad \text{--- (1)}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i$$

$$- \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^\lambda \log \left(\frac{\sigma}{x_i}\right) = 0 \quad \text{--- (2)}$$

$$(1) \Rightarrow \quad n = \sigma^\lambda \sum_{i=1}^{n} x_i^{-\lambda}$$

$$\Rightarrow \quad \sigma = \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right]^{\frac{1}{\lambda}} \quad \text{---} \quad (3)$$

Sub (3) into (2) to get

$$\frac{n}{\lambda} + \frac{n}{\lambda} \log \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right] - \sum_{i=1}^{n} \log x_i$$

$$- \left(\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right) \frac{1}{\lambda} \log \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right] \sum_{i=1}^{n} x_i^{-\lambda}$$

$$+ \left(\frac{n}{\sum_{i=1}^{n} x_i^{-\lambda}} \right) \sum_{i=1}^{n} x_i^{-\lambda} \log x_i = 0 \quad \text{---} \quad (4)$$

(4) involves only λ.

It can be solved numerically for the MLE $\hat{\lambda}$ of λ.

Sub $\hat{\lambda}$ into (3) to get

$$\hat{\sigma} = \left[\frac{n}{\sum_{i=1}^{n} x_i^{-\hat{\lambda}}} \right]^{\frac{1}{\hat{\lambda}}}.$$

Q3

$$L(\sigma,\lambda) = \prod_{i=1}^{n} \left[\lambda \sigma^{-\lambda} x_i^{\lambda-1} e^{-\left(\frac{x_i}{\sigma}\right)^{\lambda}} \right]$$

$$= \lambda^n \sigma^{-n\lambda} \left(\prod_{i=1}^{n} x_i\right)^{\lambda-1} e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}}$$

$$\log L(\sigma,\lambda) = n \log \lambda - n\lambda \log \sigma + (\lambda-1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \frac{\lambda}{\sigma^{\lambda+1}} \sum_{i=1}^{n} x_i^{\lambda} = 0 \quad \text{—— (1)}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda} \log\left(\frac{x_i}{\sigma}\right) = 0 \quad \text{—— (2)}$$

$$(1) \Rightarrow \quad \sigma = \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda}\right]^{\frac{1}{\lambda}} \quad \text{—— (3)}$$

Sub (3) into (2) to get

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda}\right] + \sum_{i=1}^{n} \log x_i$$

$$- \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda}\right]^{-1} \sum_{i=1}^{n} x_i^{\lambda} \log x_i$$

$$+ \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda}\right]^{-1} \left(\frac{1}{\lambda} \log \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda}\right]\right) \sum_{i=1}^{n} x_i^{\lambda} = 0 \quad \text{—— (4)}$$

(4) involves only λ.
It can be solved numerically for the MLE $\hat{\lambda}$ of λ. Sub $\hat{\lambda}$ into (3) to get
$$\hat{\sigma} = \left[\frac{1}{n} \sum_{i=1}^{n} x_i^{\hat{\lambda}}\right]^{1/\hat{\lambda}}.$$

Q4

$$L(\lambda) = \left[\prod_{i=1}^{n} (1 - \lambda x_i)\right]^{\frac{1}{\lambda} - 1}$$

$$\log L(\lambda) = \left(\frac{1}{\lambda} - 1\right) \sum_{i=1}^{n} \log(1 - \lambda x_i)$$

$$\frac{d \log L}{d\lambda} = -\frac{1}{\lambda^2} \sum_{i=1}^{n} \log(1 - \lambda x_i)$$

$$- \left(\frac{1}{\lambda} - 1\right) \sum_{i=1}^{n} \frac{x_i}{1 - \lambda x_i} = 0$$

$$\Rightarrow \quad - \sum_{i=1}^{n} \log(1 - \lambda x_i) = \lambda(1 - \lambda) \sum_{i=1}^{n} \frac{x_i}{1 - \lambda x_i}$$

This must be solved numerically for the MLE $\hat{\lambda}$ of λ.