EXAMPLE CLASS

19 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

$$L(\sigma) = \prod_{i=1}^{n} \left[\frac{1}{\sigma} e^{-\frac{x_i}{\sigma}} e^{-e^{-\frac{x_i}{\sigma}}} \right]$$

$$= \frac{1}{\sigma^n} e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} \; e^{-\sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}}$$

$$\log L(\sigma) = -n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^{n} x_i$$

$$- \sum_{i=1}^{n} e^{-\frac{x_i}{\sigma}}$$

$$\frac{d \log L(\sigma)}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^{n} x_i$$

$$- \sum_{i=1}^{n} \frac{x_i}{\sigma^2} e^{-\frac{x_i}{\sigma}} = 0$$

$$\Rightarrow \quad n\sigma + \sum_{i=1}^{n} x_i e^{-\frac{x_i}{\sigma}} = \sum_{i=1}^{n} x_i$$

must be solved numericalley for the MLE of σ.

Q2

$$L(\lambda,\sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{\lambda} x_i^{-\lambda-1} e^{-\sigma^{\lambda} x_i^{-\lambda}} \right]$$

$$\log L(\lambda,\sigma) = n \log \lambda + n\lambda \log \sigma$$

$$- (\lambda+1) \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma - \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^{\lambda} \log\left(\frac{\sigma}{x_i}\right) = 0 \quad \text{—— (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} - \sum_{i=1}^{n} \frac{\lambda \sigma^{\lambda-1}}{x_i^{\lambda}} = 0 \quad \text{—— (2)}$$

$$(2) \Rightarrow \sigma^{-\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda}$$

$$\Rightarrow \sigma = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda} \right)^{-\frac{1}{\lambda}} \quad \text{—— (3)}$$

Sub (3) into (1)

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda} \right) - \sum_{i=1}^{n} \log x_i$$

$$+ \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda} \right)^{-1} \sum_{i=1}^{n} x_i^{-\lambda} \log x_i$$

$$+ \frac{1}{\lambda} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda} \right)^{-1} \left[\log \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{-\lambda} \right) \right] \sum_{i=1}^{n} x_i^{-\lambda} = 0 \quad \text{—— (4)}$$

(4) only involves λ.

So it can be solved numerically for its MLE $\hat{\lambda}$.

Then sub into (3) to obtain the MLE of σ.

Q3

$$L(\lambda, \sigma) = \prod_{i=1}^{n} \left[\lambda \sigma^{-\lambda} x_i^{\lambda-1} e^{-\left(\frac{x_i}{\sigma}\right)^{\lambda}} \right]$$

$$= \lambda^n \sigma^{-n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{\lambda-1} e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}}$$

$$\log L(\lambda, \sigma) = n \log \lambda - n\lambda \log \sigma + (\lambda - 1) \sum_{i=1}^{n} \log x_i$$
$$- \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - n \log \sigma + \sum_{i=1}^{n} \log x_i$$
$$- \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^{\lambda} \log \left(\frac{x_i}{\sigma}\right) = 0 \quad \text{--- (1)}$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n\lambda}{\sigma} + \lambda \sum_{i=1}^{n} \frac{x_i^{\lambda}}{\sigma^{\lambda+1}} = 0 \quad \text{--- (2)}$$

$$(2) \Rightarrow \quad \sigma = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{\frac{1}{\lambda}} \quad \text{--- (3)}$$

Sub (3) into (1) to get

$$\frac{n}{\lambda} - \frac{n}{\lambda} \log \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right) + \sum_{i=1}^{n} \log x_i$$
$$- \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{-1} \sum_{i=1}^{n} x_i^{\lambda} \log x_i$$
$$+ \frac{1}{\lambda} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right)^{-1} \left[\log \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{\lambda} \right) \right] \frac{\cancel{\sum_{i=1}^{n} x_i^{\lambda} \log x_i}}{\left(\sum_{i=1}^{n} x_i^{\lambda} \right)} = 0 \quad \text{--- (4)}$$

(4) involves only λ.

Solve it numerically for the MLE $\hat{\lambda}$ of λ.

Sub $\hat{\lambda}$ into (3) to get the MLE $\hat{\sigma}$ of σ.