LECTURE

12 NOVEMBER

9:00-10:00AM

MATH3/4/68181

Estimation methods for VaR

a) Parametric estimation methods

b) Non-parametric " "

c) Semi-parametric " "

Level 3 – a) and b) only

Levels 4, 6 – a), b) and c)

a) Parametric estimation methods

1) Normal distribution

Suppose $X \sim N(\mu, \sigma^2)$. Then

$$VaR_p(X) = \mu + \sigma\, \Phi^{-1}(p).$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. Then the MLEs of μ and σ are

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

Hence, the MLE of $VaR_p(X)$ is

$$\widehat{VaR}_p(X) = \bar{x} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}\; \Phi^{-1}(p)$$

Suppose $\hat{\theta}$ is an estimator of θ.

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

$\hat{\theta}$ is unbiased if $\text{Bias}(\hat{\theta}) = 0$

$$\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$$

Mean Squared Error

$\hat{\theta}$ is consistent if $\text{MSE}(\hat{\theta}) \to 0$ as $n \to \infty$.

$\hat{\theta}$ is asymptotically unbiased if $\text{Bias}(\hat{\theta}) \to 0$ as $n \to \infty$

$\widehat{\text{VaR}_p}(X)$ is a biased estimator.

But it is consistent.

2) Uniform distribution method

Suppose $X \sim Uni[a, b]$. Then

$$VaR_p(X) = a + p(b-a)$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. Then the MLEs of a and b are

$$\hat{a} = \min(X_1, \ldots, X_n)$$

$$\hat{b} = \max(X_1, \ldots, X_n).$$

Hence the MLE of $VaR_p(X)$ is

$$\widehat{VaR}_p(X) = \min(X_1, \ldots, X_n)$$

$$+ p\left[\max(X_1, \ldots, X_n) - \min(X_1, \ldots, X_n)\right].$$

Homework : $\widehat{VaR}_p(X)$ is a biased estimator but consistent

3) Power function distribution method

Suppose X has CDF $F(x) = x^a$

Then

$$VaR_p(X) = p^{\frac{1}{a}}$$

Suppose $x_1, x_2, \dots, x_n$ is a random sample on X. The likelihood function of a is

$$L(a) = \prod_{i=1}^{n} \left[a X_i^{a-1}\right]$$

$$= a^n \left(\prod_{i=1}^{n} X_i\right)^{a-1}$$

The log-likelihood function is

$$\log L(a) = n \log a + (a-1) \sum_{i=1}^{n} \log X_i .$$

The derivative wrt a is

$$\frac{d \log L(a)}{da} = \frac{n}{a} + \sum_{i=1}^{n} \log X_i .$$

Setting this to zero, we obtain

$$\hat{a} = - \frac{n}{\sum_{i=1}^{n} \log X_i} .$$

This is an MLE since

$$\frac{d^2 \log L(a)}{da^2} = - \frac{n}{a^2} < 0 \quad \forall a$$

Hence, $\hat{a}$ is the MLE of a. The MLE of $VaR_p(X)$ is

$$\widehat{VaR_p(X)} = p^{-\frac{\sum_{i=1}^{n} \log X_i}{n}}$$

4) Variance - Covariance method (indep case)

Let

$$X = \sum_{j=1}^{p} w_j X_j$$

no of investments

weighted portfolio loss

loss for jth investment

weight for jth investment

Suppose $X_j \sim N(\mu_j, \sigma_j^2)$ independently.

Then

$$X \sim N\left(\sum_{j=1}^{p} w_j \mu_j, \sum_{j=1}^{p} w_j^2 \sigma_j^2\right).$$

So,

$$VaR_p(X) = \sum_{j=1}^{p} w_j \mu_j + \Phi^{-1}(p) \sqrt{\sum_{j=1}^{p} w_j^2 \sigma_j^2}.$$

Suppose $x_{j1}, x_{j2}, \ldots, x_{jn}$ is a random sample on X_j. Then the MLEs of μ_j and σ_j^2 are

$$\widehat{\mu_j} = \frac{1}{n} \sum_{k=1}^{n} x_{jk}$$

and

$$\widehat{\sigma_j^2} = \frac{1}{n} \sum_{k=1}^{n} \left(x_{jk} - \widehat{\mu_j}\right)^2.$$

Hence, the MLE of $VaR_p(X)$ is

$$\widehat{VaR_p(X)} = \sum_{j=1}^{p} w_j \widehat{\mu_j} + \Phi^{-1}(p) \sqrt{\sum_{j=1}^{p} w_j^2 \widehat{\sigma_j^2}}$$

5) Weibull distribution method.

Suppose X has the CDF

$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}}, \quad x > 0.$$

To find VaR, set

$$F(x) = p$$

$$\Rightarrow \quad 1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}} = p$$

$$\Rightarrow \quad e^{-\left(\frac{x}{\theta}\right)^{\beta}} = 1-p$$

$$\Rightarrow \quad \left(\frac{x}{\theta}\right)^{\beta} = -\log(1-p)$$

$$\Rightarrow \quad x = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

$$\Rightarrow \quad VaR_p(X) = \theta\left[-\log(1-p)\right]^{\frac{1}{\beta}}$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. It can be shown that the estimates of θ and β are the solutions of

$$\frac{\bar{x}^2}{s^2} = \frac{\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2}{\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2}$$

and

$$\hat{\theta} = \frac{\bar{x}}{\Gamma\left(1+\frac{1}{\hat{\beta}}\right)}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

Hence the estimate of VaR is

$$\widehat{VaR}_p(X) = \hat{\theta}\left[-\log(1-p)\right]^{\frac{1}{\hat{\beta}}}$$

b) Non-parametric estimation methods

1) Historical method

Data $-$ $x_1, x_2, \dots, x_n$

Order the data from smallest to largest

$$\boxed{x_{(1)}} \leq x_{(2)} \leq \cdots \leq \boxed{x_{(n)}}$$

smallest largest

Then
$$\widehat{VaR}_p(X) = x_{(i)}$$
if $p \in \left(\frac{i-1}{n}, \frac{i}{n} \right]$.

Ex X = Losses

Data on losses as

8 -2 2 0 5

Order the data

$$\begin{array}{ccccc} -2 & 0 & 2 & 5 & 8 \\ \| & \| & \| & \| & \| \\ x_{(1)} & x_{(2)} & x_{(3)} & x_{(4)} & x_{(5)} \end{array}$$

$$\widehat{VaR}_{0.2}(X) = x_{(1)} = -2$$

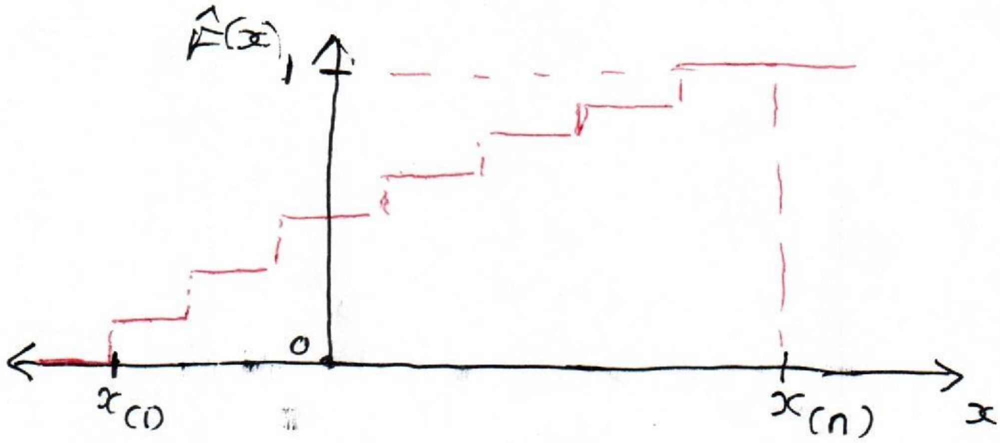
$$\widehat{VaR}_{0.9}(X) = x_{(5)} = 8$$

2) Bootstrap method due to Efron

Data $x_1, x_2, \dots, x_n$

Compute the empirical CDF

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} I\{x_j \leqslant x\}$$



i) simulate B samples from $\hat{F}$

ii) estimate VaR_p for each of the samples by the historical method, yielding $\widehat{VaR}_p^{(j)}$, $j = 1, 2, \dots, B$

iii) Compute $\widehat{VaR}_p$ as either

$$\widehat{VaR}_p = \text{Mean}\left(\widehat{VaR}_p^{(1)}, \dots, \widehat{VaR}_p^{(B)}\right)$$

or

$$\widehat{VaR}_p = \text{Median}\left(\widehat{VaR}_p^{(1)}, \dots, VaR_p^{(B)}\right)$$

3) Jackknife method

Data $x_1, x_2, \ldots, x_n$

(i) estimate VaR_p by the historical method for $x_2, x_3, \ldots, x_n$.
Call this $\widehat{VaR}_p^{(1)}$.

(ii) estimate VaR_p by the historical method for $x_1, x_3, \ldots, x_n$.
Call this $\widehat{VaR}_p^{(2)}$.

⋮

(iii) estimate VaR_p by the historical method for $x_1, x_2, \ldots, x_{n-1}$.
Call this $\widehat{VaR}_p^{(n)}$

(iv) Compute $\widehat{VaR}_p$ as either

$$\widehat{VaR}_p = \text{Mean}\left(\widehat{VaR}_p^{(1)}, \ldots, \widehat{VaR}_p^{(n)}\right)$$

or

$$\widehat{VaR}_p = \text{Median}\left(\widehat{VaR}_p^{(1)}, \ldots, \widehat{VaR}_p^{(n)}\right).$$

4) Kernel method