EXAMPLE CLASS

13 NOVEMBER

16:00-17:00PM

MATH3/4/68181

Q1

$$X \sim \text{Exp}(\lambda)$$

From example class last week,

$$VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

and

$$ES_p(X) = -\frac{1}{p\lambda}\left[p\log(1-p) - p - \log(1-p)\right].$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from $\text{Exp}(\lambda)$. The MLE of λ is

$$\hat{\lambda} = \frac{1}{\bar{x}} \quad (\text{from Math } 20802).$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = -\bar{x}\log(1-p)$$

and

$$\widehat{ES}_p(X) = -\frac{\bar{x}}{\lambda}\left[p\log(1-p) - p - \log(1-p)\right]$$

Q2

X has PDF $f(x) = a x^{a-1}$

The CDF $F(x) = x^{a}$, $0 < x < 1$

From Monday's lecture,

$$\mathrm{VaR}_p(X) = p^{\frac{1}{a}},$$

and

$$\mathrm{ES}_p(X) = \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}.$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. The MLE of a is

$$\widehat{a} = -\frac{n}{\sum_{i=1}^{n} \log X_i}.$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{\mathrm{VaR}}_p(X) = p^{\frac{1}{\widehat{a}}}$$

and

$$\widehat{\mathrm{ES}}_p(X) = \frac{p^{\frac{1}{\widehat{a}}}}{\frac{1}{\widehat{a}}+1}.$$

Q3

covered in lectures

Q4

$X \sim LN(\mu, \sigma^2)$

The CDF of X is

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{\log x - \mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow \log x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow x = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$\Rightarrow VaR_p(X) = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$\Rightarrow ES_p(X) = \frac{1}{p} \int_0^p e^{\mu + \sigma \Phi^{-1}(t)} dt$$

$$= \frac{e^{\mu}}{p} \int_0^p e^{\sigma \Phi^{-1}(t)} dt$$

From Math 20802, the MLEs of μ and σ are

$$\widehat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log x_i$$

and $$\widehat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\log x_i - \widehat{\mu})^2}$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = e^{\widehat{\mu} + \widehat{\sigma} \Phi^{-1}(p)},$$

$$\widehat{ES}_p(X) = \frac{e^{\widehat{\mu}}}{p} \int_0^p e^{\widehat{\sigma} \Phi^{-1}(t)} dt$$

Q5

X has PDF $f(x) = \theta_2 x^{\theta_2 - 1} \theta_1^{-\theta_2}$,

$0 < x < \theta_1$

The CDF $F(x) = \left(\frac{x}{\theta_1}\right)^{\theta_2} = p$

$$\Rightarrow \quad x = \theta_1 \, p^{\frac{1}{\theta_2}}$$

$$\Rightarrow \quad VaR_p(X) = \theta_1 \, p^{\frac{1}{\theta_2}}$$

$$\Rightarrow \quad ES_p(X) = \frac{\theta_1}{p} \int_0^p t^{\frac{1}{\theta_2}} \, dt$$

$$= \frac{\theta_1}{1 + \frac{1}{\theta_2}} \, p^{\frac{1}{\theta_2}}$$

From Math 20802, the MLEs of θ_1 and θ_2 are

$$\widehat{\theta_1} = \max(x_1, x_2, \ldots, x_n)$$

and $$\widehat{\theta_2} = \frac{n}{n \log \widehat{\theta_1} - \sum_{i=1}^{n} \log x_i}.$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR_p} = \widehat{\theta_1} \, p^{\frac{1}{\widehat{\theta_2}}},$$

$$\widehat{ES_p} = \frac{\widehat{\theta_1} \, \widehat{\theta_2}}{\widehat{\theta_2} + 1} \, p^{\frac{1}{\widehat{\theta_2}}}$$