EXAMPLE CLASS

12 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

$$X \sim Exp(\lambda)$$

$$\begin{cases} VaR_p(X) = -\frac{1}{\lambda}\log(1-p) \\ ES_p(X) = -\frac{1}{p\lambda}\{p\cdot\log(1-p) - p - \log(1-p)\} \end{cases}$$

From example class last week

Suppose $x_1, x_2, \dots, x_n$ is a random sample from $Exp(\lambda)$.

The MLE of

$$\hat{\lambda} = \frac{1}{\bar{x}} \quad (\text{from Math20802}).$$

Hence MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = -\bar{x}\cdot\log(1-p)$$

and $$\widehat{ES_p}(X) = -\frac{\bar{x}}{p}\{p\cdot\log(1-p) - p - \log(1-p)\}$$

Q2

X has PDF $a x^{a-1}$

$$\mathrm{VaR}_p(X) = p^{\frac{1}{a}}$$

$$\mathrm{ES}_p(X) = \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}$$

from Example class last week.

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. The MLE of a is

$$\hat{a} = -\frac{n}{\sum_{i=1}^{n} \log x_i} \quad \left(\begin{array}{l}\text{from lecture} \\ \text{this morning}\end{array}\right)$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{\mathrm{VaR}_p(X)} = p^{-\frac{\sum_{i=1}^{n} \log x_i}{n}}$$

and

$$\widehat{\mathrm{ES}_p(X)} = \frac{p^{-\frac{\sum_{i=1}^{n} \log x_i}{n}}}{-\frac{\sum_{i=1}^{n} \log x_i}{n} + 1}$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = e^{\hat{\mu} + \hat{\sigma}\Phi^{-1}(p)}$$

and

$$\widehat{ES}_p(X) = \frac{e^{\hat{\mu}}}{p}\int_0^p e^{\hat{\sigma}\Phi^{-1}(t)}\,dt.$$

Q3 Covered in lectures

Q4

$X \sim LN(\mu, \sigma^2)$

The CDF of X is

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right) = p$$

$$\Rightarrow \frac{\log x - \mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow \log x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow x = e^{\mu + \sigma \Phi^{-1}(p)}$$

$$\Rightarrow \boxed{VaR_p(X) = e^{\mu + \sigma \Phi^{-1}(p)}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p e^{\mu + \sigma \Phi^{-1}(t)}\, dt$$

$$= \boxed{\frac{e^{\mu}}{p}\int_0^p e^{\sigma \Phi^{-1}(t)}\, dt}$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from $LN(\mu, \sigma^2)$. The MLEs of μ and σ are

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \log x_i$$

and

$$\widehat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\log x_i - \widehat{\mu}\right)^2}$$

(from Math 20802)

Q5

X has PDF $f(x) = \theta_2 x^{\theta_2 - 1} \theta_1^{-\theta_2}$,
$0 < x < \theta_1$

The CDF of X is

$$F(x) = \theta_2 \theta_1^{-\theta_2} \int_0^x y^{\theta_2 - 1} \, dy$$

$$= \theta_2 \theta_1^{-\theta_2} \left[\frac{y^{\theta_2}}{\theta_2} \right]_0^x$$

$$= \left(\frac{x}{\theta_1} \right)^{\theta_2}$$

To find VaR, set

$$\left(\frac{x}{\theta_1} \right)^{\theta_2} = p$$

$$\Rightarrow \quad x = \theta_1 p^{\frac{1}{\theta_2}}$$

$$\Rightarrow \quad VaR_p(X) = \boxed{\theta_1 p^{\frac{1}{\theta_2}}}$$

$$\Rightarrow \quad ES_p(X) = \frac{\theta_1}{p} \int_0^p t^{\frac{1}{\theta_2}} \, dt$$

$$= \frac{\theta_1}{p} \left[\frac{t^{\frac{1}{\theta_2} + 1}}{\frac{1}{\theta_2} + 1} \right]_0^p$$

$$= \boxed{\frac{\theta_1 \theta_2}{1 + \theta_2} p^{1/\theta_2}}$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. The MLEs of θ_1 and θ_2 are

$$\widehat{\theta}_1 = \max(x_1, x_2, \ldots, x_n)$$

and
$$\widehat{\theta_2} = \frac{n}{n \log \widehat{\theta}_1 - \sum_{i=1}^{n} \log x_i}.$$

(from Math20802)

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR_p}(X) = \widehat{\theta_1}\, p^{\frac{1}{\widehat{\theta_2}}}$$

and
$$\widehat{ES_p}(X) = \frac{\widehat{\theta_1}\,\widehat{\theta_2}}{1+\widehat{\theta_2}}\, p^{\frac{1}{\widehat{\theta_2}}}.$$

Q6

$$X \sim \text{Uni}\,[\mu - \delta, \ \mu + \delta]$$

$$F(x) = \frac{x - \mu + \delta}{2\delta} = p$$

$$\Rightarrow \quad VaR_p(X) = \mu - \delta + 2\delta p$$

$$\Rightarrow \quad ES_p(X) = \mu - \delta + \delta p$$

From Math 20802, the MLEs of μ and σ are

$$\widehat{\mu} = \frac{\max x_i + \min x_i}{2}$$

$$\widehat{\delta} = \frac{\max x_i - \min x_i}{2}.$$

Hence, the MLEs of VaR_p and ES_p are

$$\widehat{VaR}_p(X) = \widehat{\mu} - \widehat{\delta} + 2\widehat{\delta}\, p,$$

$$\widehat{ES}_p(X) = \widehat{\mu} - \widehat{\delta} + \widehat{\delta}\, p.$$