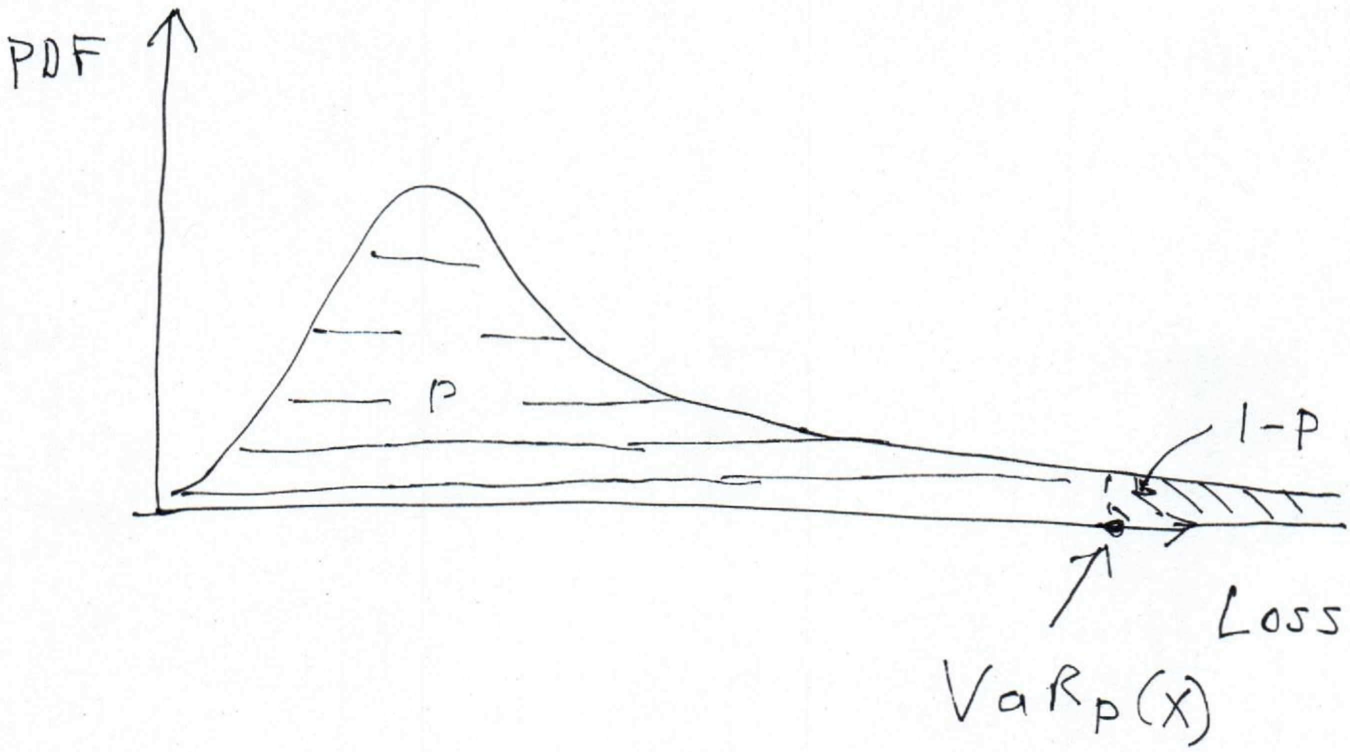
LECTURE

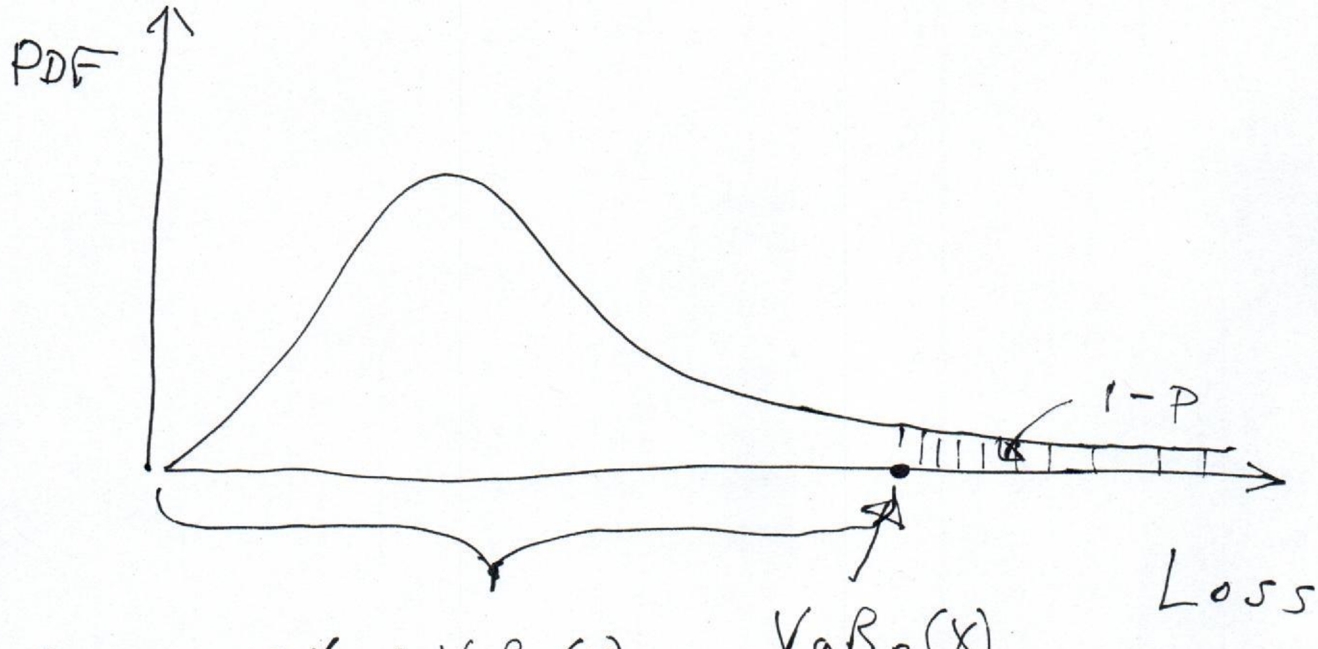
6 NOVEMBER

9:00-10:00AM

MATH3/4/68181



PDF
p
1-p
Loss
VaR_p(X)



Average of $X \leqslant VaR_p(X)$

$=$

$ES_p(X)$

Ex1 Suppose $X \sim N(\mu, \sigma^2)$

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = p$$

$$\Rightarrow \quad \frac{x-\mu}{\sigma} = \Phi^{-1}(p)$$

$$\Rightarrow \quad x = \mu + \sigma \Phi^{-1}(p)$$

$$\Rightarrow \quad VaR_p(X) = \mu + \sigma \Phi^{-1}(p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \left[\mu + \sigma \Phi^{-1}(t)\right] dt$$

$$= \mu + \frac{\sigma}{p}\int_0^p \Phi^{-1}(t)\, dt$$

Ex 2

$$X \sim \text{Uni}[a, b]$$

$$F(x) = \frac{x-a}{b-a} = p$$

$$\Rightarrow \quad x = a + p(b-a)$$

$$\Rightarrow \quad VaR_p(X) = a + p(b-a)$$

$$ES_p(X) = \frac{1}{p}\int_0^p [a + t(b-a)]\, dt$$

$$= \frac{1}{p}\left[at + \frac{t^2}{2}(b-a)\right]_0^p$$

$$= a + \frac{p}{2}(b-a)$$

Ex 3

Suppose X has CDF $F(x) = x^a$, $0 < x < 1$

$$F(x) = x^a = p$$

$$\Rightarrow x = p^{\frac{1}{a}}$$

$$\Rightarrow VaR_p(X) = p^{\frac{1}{a}}$$

$$ES_p(X) = \frac{1}{p} \int_0^p t^{\frac{1}{a}} dt$$

$$= \frac{1}{p} \left[\frac{t^{\frac{1}{a}+1}}{\frac{1}{a}+1} \right]_0^p$$

$$= \frac{p^{\frac{1}{a}}}{\frac{1}{a}+1}.$$

Proof of Properties of VaR

Translative property

$$VaR_p(X+c) = VaR_p(X) + c \qquad (*)$$

Proof. Suppose $(*)$ holds. Then

$$F^{-1}_{X+c}(p) = F^{-1}_{X}(p) + c$$

$$\Rightarrow \quad F_{X+c}\left(F^{-1}_{X+c}(p)\right) = F_{X+c}\left(F^{-1}_{X}(p) + c\right)$$

$$\Rightarrow \quad p = F_{X+c}\left(F^{-1}_{X}(p) + c\right)$$

$$\Rightarrow \quad p = Pr\left(X + \cancel{c} \leqslant F^{-1}_{X}(p) + \cancel{c}\right)$$

$$\Rightarrow \quad p = Pr\left(X \leqslant F^{-1}_{X}(p)\right)$$

$$\Rightarrow \quad p = F_X\left(F^{-1}_{X}(p)\right)$$

$$\Rightarrow \quad p = p.$$

Hence $(*)$ must hold.

Monotone property

$$X \leq Y \Rightarrow VaR_p(X) \leq VaR_p(Y)$$

<u>Proof</u> $\quad X \leq Y$

$$\Rightarrow F_X^{-1}(p) \leq F_Y^{-1}(p) \quad \forall p$$

$$\Rightarrow VaR_p(X) \leq VaR_p(Y) \quad \forall p.$$

Positive homogeneity

$$VaR_p(cX) = c\, VaR_p(X) \ldots \quad (*)$$

Proof Suppose $(*)$ holds. Then

$$F_{cX}^{-1}(p) = c\, F_X^{-1}(p)$$

$$\Rightarrow \quad F_{cX}\left(F_{cX}^{-1}(p)\right) = F_{cX}\left(c F_X^{-1}(p)\right)$$

$$\Rightarrow \quad p = F_{cX}\left(c F_X^{-1}(p)\right)$$

$$\Rightarrow \quad p = Pr\left(\cancel{c}X \leq \cancel{c} F_X^{-1}(p)\right)$$

$$\Rightarrow \quad p = F_X\left(F_X^{-1}(p)\right)$$

$$\Rightarrow \quad p = p.$$

Hence $(*)$ must hold.

Estimation methods for VaR

a) Parametric estimation methods

b) Non-parametric " "

c) Semi-parametric " "

Level 3 — a) and b) only

Levels 4, 6 — a), b) and c)

a) Parametric estimation methods

1) Normal distribution

Suppose $X \sim N(\mu, \sigma^2)$. Then

$$VaR_p(X) = \mu + \sigma \Phi^{-1}(p).$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. Then the MLEs of μ and σ are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}.$$

Hence, the MLE of $VaR_p(X)$ is

$$\widehat{VaR}_p(X) = \bar{x} + \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}\, \Phi^{-1}(p)$$

Suppose $\hat{\theta}$ is an estimator of θ.

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

$\hat{\theta}$ is unbiased if $\text{Bias}(\hat{\theta}) = 0$

$$\text{MSE}(\hat{\theta}) = E\left[(\hat{\theta} - \theta)^2\right]$$

Mean Squared Error

$\hat{\theta}$ is consistent if $\text{MSE}(\hat{\theta}) \to 0$ as $n \to \infty$.

$\hat{\theta}$ is asymptotically unbiased if $\text{Bias}(\hat{\theta}) \to 0$ as $n \to \infty$

$\widehat{\text{VaR}_p}(X)$ is a biased estimator.

But it is consistent.

2) Uniform distribution method

Suppose $X \sim \mathrm{Uni}[a, b]$. Then

$$\mathrm{VaR}_p(X) = a + p(b-a)$$

Suppose $x_1, x_2, \ldots, x_n$ is a random sample on X. Then the MLEs of a and b are

$$\hat{a} = \min(X_1, \ldots, X_n)$$

$$\hat{b} = \max(X_1, \ldots, X_n).$$

Hence the MLE of $\mathrm{VaR}_p(X)$ is

$$\widehat{\mathrm{VaR}_p}(X) = \min(X_1, \ldots, X_n) + p\left[\max(X_1, \ldots, X_n) - \min(X_1, \ldots, X_n)\right].$$

Homework : $\widehat{\mathrm{VaR}}_p(X)$ is a biased estimator but consistent