EXAMPLE CLASS

6 NOVEMBER

16:00-17:00PM

MATH3/4/68181

Q1

$$F(x) = 1 - e^{-\lambda x} = p$$

$$\Rightarrow \quad e^{-\lambda x} = 1 - p$$

$$\Rightarrow \quad -\lambda x = \log(1-p)$$

$$\Rightarrow \quad x = -\frac{1}{\lambda} \log(1-p)$$

$$\Rightarrow \quad VaR_p(X) = -\frac{1}{\lambda} \log(1-p)$$

$$ES_p(X) = \frac{1}{p} \int_0^p \left(-\frac{1}{\lambda}\right) \log(1-t)\, dt$$

$$= -\frac{1}{\lambda p} \int_0^p \log(1-t)\, dt$$

$$= -\frac{1}{\lambda p} \left\{ \left[t \cdot \log(1-t) \right]_0^p - \int_0^p t \frac{(-1)}{1-t} dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) - 0 + \int_0^p \frac{t-1+1}{1-t} dt \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) + \left[-t - \log(1-t) \right]_0^p \right\}$$

$$= -\frac{1}{\lambda p} \left\{ p \cdot \log(1-p) - p - \log(1-p) \right\}$$

Q4

$$F(x) = 1 - \left(\frac{K}{x}\right)^{a}, \quad x > K$$

$$= p$$

$$\Rightarrow \left(\frac{K}{x}\right)^{a} = 1 - p$$

$$\Rightarrow \frac{K}{x} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow x = K(1-p)^{-\frac{1}{a}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p K(1-t)^{-\frac{1}{a}}\,dt$$

$$= \frac{K}{p}\left[\frac{(1-t)^{1-\frac{1}{a}}}{\left(1-\frac{1}{a}\right)(-1)}\right]_0^p$$

$$= \frac{K}{p}\ \frac{(1-p)^{1-\frac{1}{a}} - 1}{\frac{1}{a} - 1}$$

Q6

$$F(x) = \frac{1}{1 + \left(\frac{x}{a}\right)^{-b}} = p$$

$$\Rightarrow \quad 1 + \left(\frac{x}{a}\right)^{-b} = \frac{1}{p}$$

$$\Rightarrow \quad \left(\frac{x}{a}\right)^{-b} = \frac{1-p}{p}$$

$$\Rightarrow \quad x = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p a\left(\frac{1-t}{t}\right)^{-\frac{1}{b}} dt$$

$$= \frac{a}{p}\int_0^p t^{\frac{1}{b}}(1-t)^{-\frac{1}{b}}\, dt$$

$$= \frac{a}{p} B_p\left(1+\frac{1}{b},\ 1-\frac{1}{b}\right)$$

$$\boxed{B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\, dt}$$

Incomplete Beta Function

Q7

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = p$$

$$\Rightarrow \left(1 + \frac{x}{\lambda}\right)^{-\alpha} = 1 - p$$

$$\Rightarrow 1 + \frac{x}{\lambda} = (1-p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow x = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right] = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \lambda\left[(1-t)^{-\frac{1}{\alpha}} - 1\right] dt$$

$$= \frac{\lambda}{p}\left[\frac{(1-t)^{1-\frac{1}{\alpha}}}{\left(1-\frac{1}{\alpha}\right)(-1)} - t\right]_0^p$$

$$= \frac{\lambda}{p}\left[\frac{(1-p)^{1-\frac{1}{\alpha}}}{\left(\frac{1}{\alpha}-1\right)} - p - \frac{1}{\left(\frac{1}{\alpha}-1\right)}\right].$$

Q 8

$$F(x) = e^{-\left(\frac{\sigma}{x}\right)^{\alpha}} = p$$

$$\Rightarrow \quad -\left(\frac{\sigma}{x}\right)^{\alpha} = \log p$$

$$\Rightarrow \quad \frac{\sigma}{x} = (-\log p)^{\frac{1}{\alpha}}$$

$$\Rightarrow \quad x = \sigma(-\log p)^{-\frac{1}{\alpha}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p \sigma(-\log t)^{-\frac{1}{\alpha}}\,dt$$

$$= \frac{\sigma}{p}\int_0^p (-\log t)^{-\frac{1}{\alpha}}\,dt$$

$$\boxed{\text{Set } \; y = -\log t \Rightarrow t = e^{-y} \Rightarrow \frac{dt}{dy} = -e^{-y}}$$

$$= \frac{\sigma}{p}\int_{\infty}^{-\log p} y^{-\frac{1}{\alpha}}\left(-e^{-y}\right)dy$$

$$= \frac{\sigma}{p}\int_{-\log p}^{\infty} y^{-\frac{1}{\alpha}} e^{-y}\,dy$$

$$= \frac{\sigma}{p}\Gamma\left(1-\frac{1}{\alpha}, -\log p\right)$$

$$\boxed{\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t}\,dt}$$

Lower Incomplete gamma function

Q9

$$F(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^{\alpha}} = p$$

$$\Rightarrow \quad x = \sigma \left[-\log(1-p) \right]^{\frac{1}{\alpha}}$$

$$= VaR_p(X)$$

$$ES_p(X) = \frac{\sigma}{p} \int_0^p \left[-\log(1-t) \right]^{\frac{1}{\alpha}} dt$$

$$\boxed{\begin{aligned} \text{Set} \quad y = -\log(1-t) \Rightarrow \ & 1 - t = e^{-y} \\ \Rightarrow \ & t = 1 - e^{-y} \\ \Rightarrow \ & \frac{dt}{dy} = e^{-y} \end{aligned}}$$

$$= \frac{\sigma}{p} \int_0^{-\log(1-p)} y^{\frac{1}{\alpha}} e^{-y} \, dy$$

$$= \frac{\sigma}{p} \, \gamma\left(1 + \frac{1}{\alpha}, -\log(1-p)\right)$$

$$\boxed{\gamma(a, x) = \int_0^x t^{a-1} e^{-t} \, dt}$$

Upper incomplete gamma function