EXAMPLE CLASS

6 NOVEMBER

12:00-13:00PM

MATH3/4/68181

Q1

$$F(x) = 1 - e^{-\lambda x} = p$$

$$\boxed{\begin{aligned} VaR_p(X) &= F^{-1}(p) \\ ES_p(X) &= \frac{1}{p}\int_0^p VaR_t(X)\,dt \end{aligned}}$$

$$1 - e^{-\lambda x} = p$$

$$\Rightarrow \quad e^{-\lambda x} = 1-p$$

$$\Rightarrow \quad \lambda x = -\log(1-p)$$

$$\Rightarrow \quad x = -\frac{1}{\lambda}\log(1-p)$$

$$\Rightarrow \quad VaR_p(X) = -\frac{1}{\lambda}\log(1-p)$$

$$ES_p(X) = \frac{1}{p}\int_0^p -\frac{1}{\lambda}\log(1-t)\,dt$$

$$= -\frac{1}{\lambda p}\int_0^p \log(1-t)\,dt$$

$$= -\frac{1}{\lambda p}\left\{\Big[t \cdot \log(1-t)\Big]_0^p - \int_0^p t \cdot \frac{(-1)}{1-t}\,dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{p\log(1-p) - 0 + \int_0^p \frac{(t-1)+1}{1-t}\,dt\right\}$$

$$= -\frac{1}{\lambda p}\left\{ p \log(1-p) + \int_0^p \left(-1 + \frac{1}{1-t}\right) dt \right\}$$

$$= -\frac{1}{\lambda p}\left\{ p \log(1-p) + \Big[-t - \log(1-t)\Big]_0^p \right\}$$

$$= -\frac{1}{\lambda p}\left\{ p \log(1-p) - p - \log(1-p) \right\}$$

$$\Rightarrow \quad ES_p(X) = -\frac{1}{\lambda p}\left\{ (p-1)\log(1-p) - p \right\}.$$

Q2

$$F(x) = x^{\alpha} = p$$

$$\Rightarrow \quad x = p^{\frac{1}{\alpha}}$$

$$\Rightarrow \quad VaR_p(X) = p^{\frac{1}{\alpha}}$$

$$ES_p(X) = \frac{1}{p}\int_0^p t^{\frac{1}{\alpha}}\,dt$$

$$= \frac{1}{p}\left[\frac{t^{\frac{1}{\alpha}+1}}{\frac{1}{\alpha}+1}\right]_0^p$$

$$= \frac{p^{\frac{1}{\alpha}}}{\frac{1}{\alpha}+1}.$$

Q3

$$F(x) = \frac{x-a}{b-a} = p$$

$$\Rightarrow \quad x = a + (b-a)p$$

$$\Rightarrow \quad VaR_p(X) = a + (b-a)p$$

$$ES_p(X) = \frac{1}{p}\int_0^p \left[a + (b-a)t\right] dt$$

$$= \frac{1}{p}\left[at + (b-a)\frac{t^2}{2}\right]_0^p$$

$$= \frac{1}{p}\left[ap + (b-a)\frac{p^2}{2} - 0\right]$$

$$= a + (b-a)\frac{p}{2}.$$

Q4

$$F(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x > k$$

$$= p$$

$$\Rightarrow \quad \left(\frac{k}{x}\right)^a = 1 - p$$

$$\Rightarrow \quad \frac{k}{x} = (1-p)^{\frac{1}{a}}$$

$$\Rightarrow \quad x = k\,(1-p)^{-\frac{1}{a}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p k\,(1-t)^{-\frac{1}{a}}\,dt$$

$$= \frac{k}{p}\left[\frac{(1-t)^{1-\frac{1}{a}}}{\left(1-\frac{1}{a}\right)(-1)}\right]_0^p$$

$$= \frac{k}{p\left(\frac{1}{a}-1\right)}\left[(1-p)^{1-\frac{1}{a}} - 1\right].$$

Q5

$$F(x) = \Phi(x)$$

CDF of $N(0,1)$

$$= p$$

$$\Rightarrow \quad x = \Phi^{-1}(p) = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \Phi^{-1}(t)\,dt .$$

Q6

$$F(x) = \frac{1}{1+\left(\frac{x}{a}\right)^{-b}} = p$$

$$\Rightarrow \quad 1+\left(\frac{x}{a}\right)^{-b} = \frac{1}{p}$$

$$\Rightarrow \quad \left(\frac{x}{a}\right)^{-b} = \frac{1-p}{p}$$

$$\Rightarrow \quad x = a\left(\frac{1-p}{p}\right)^{-\frac{1}{b}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p a\cdot\left(\frac{1-t}{t}\right)^{-\frac{1}{b}} dt$$

$$= \frac{a}{p}\int_0^p t^{\frac{1}{b}}(1-t)^{-\frac{1}{b}} dt$$

$$= \frac{a}{p} B_p\left(1+\frac{1}{b},\ 1-\frac{1}{b}\right)$$

$$\boxed{B_p(\alpha,\beta) = \int_0^p t^{\alpha-1}(1-t)^{\beta-1} dt}$$

Incomplete beta function

Q7

$$F(x) = 1 - \left(1+\frac{x}{\lambda}\right)^{-\alpha} = p$$

$$\Rightarrow \left(1+\frac{x}{\lambda}\right)^{-\alpha} = 1-p$$

$$\Rightarrow 1+\frac{x}{\lambda} = (1-p)^{-\frac{1}{\alpha}}$$

$$\Rightarrow x = \lambda\left[(1-p)^{-\frac{1}{\alpha}} - 1\right] = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \lambda\left[(1-t)^{-\frac{1}{\alpha}} - 1\right] dt$$

$$= \frac{\lambda}{p}\left\{\frac{(1-t)^{1-\frac{1}{\alpha}}}{\left(1-\frac{1}{\alpha}\right)(-1)} - t\right\}_0^p$$

$$= \frac{\lambda}{p}\left\{\frac{(1-p)^{1-\frac{1}{\alpha}}}{-\left(1-\frac{1}{\alpha}\right)} - p + \frac{1}{1-\frac{1}{\alpha}}\right\}.$$

Q8

$$F(x) = e^{-\left(\frac{\sigma}{x}\right)^{\alpha}} = p$$

$$\Rightarrow -\left(\frac{\sigma}{x}\right)^{\alpha} = \log p$$

$$\Rightarrow \frac{\sigma}{x} = (-\log p)^{\frac{1}{\alpha}}$$

$$\Rightarrow x = \sigma(-\log p)^{-\frac{1}{\alpha}} = VaR_p(X)$$

$$ES_p(X) = \frac{1}{p}\int_0^p \sigma(-\log t)^{-\frac{1}{\alpha}}\,dt$$

$$= \frac{\sigma}{p}\int_0^p (-\log t)^{-\frac{1}{\alpha}}\,dt$$

$$\boxed{\text{Set } y = -\log t \Rightarrow t = e^{-y} \Rightarrow \frac{dt}{dy} = -e^{-y}}$$

$$= \frac{\sigma}{p}\int_{+\infty}^{-\log p} y^{-\frac{1}{\alpha}}\left(-e^{-y}\right)dy$$

$$= -\frac{\sigma}{p}\int_{+\infty}^{-\log p} y^{-\frac{1}{\alpha}} e^{-y}\,dy = \frac{\sigma}{p}\int_{-\log p}^{\infty} y^{-\frac{1}{\alpha}} e^{-y}\,dy$$

$$= \cancel{-\frac{\sigma}{p}\,\gamma\left(1-\frac{1}{\alpha}, -\log p\right)} = \frac{\sigma}{p}\Gamma\left(1-\frac{1}{\alpha}, -\log p\right)$$

$$\boxed{\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t}\,dt}$$

Incomp gamma Function.