LECTURE

26 OCTOBER

10:00-11:00AM

MATH4/68181

Case iv $X_1, \dots, X_N$ are INID, N is a RV independent of $X_1, X_2, \dots$

1) $S = X_1 + \dots + X_N$

The CDF of S is

$$F_S(s) = P[X_1 + \dots + X_N \leq s]$$

$$= \sum_{n=1}^{\infty} P[X_1 + \dots + X_N \leq s \mid N=n] \, P(N=n)$$

Total Probability Rule

$$= \sum_{n=1}^{\infty} \left[\int \dots \int_{x_1 + \dots + x_n \leq s} F_n(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \dots dx_1 \right] P(N=n)$$

The PDF of S is

$$f_S(s) = \sum_{n=1}^{\infty} \left[\int \dots \int_{x_1 + \dots + x_n = s} f_n(s - x_1 - \dots - x_{n-1}) \prod_{j=1}^{n-1} f_j(x_j) \, dx_{n-1} \, dx_{n-2} \dots dx_1 \right] P(N=n)$$

The m^{th} moment of S is

$$E[S^m] = E[(X_1 + \dots + X_N)^m]$$

$$= \sum_{n=1}^{\infty} E[(X_1 + \dots + X_N)^m \mid N=n] \, P(N=n)$$

$$= \sum_{n=1}^{\infty} \sum_{p_1 + \dots + p_n = m} \binom{m}{p_1 \, p_2 \dots p_n} E[X_1^{p_1}] \dots E[X_n^{p_n}] \, P(N=n)$$

The mean of S is

$$E[S] = \sum_{n=1}^{\infty} E[S|N=n] \; P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{j=1}^{n} E(X_j) \right] P(N=n)$$

The variance of S is

$$Var[S] = \sum_{n=1}^{\infty} Var[S|N=n] \; P(N=n)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{j=1}^{n} Var(X_j) \right] P(N=n)$$

2) $U = \max(X_1, X_2, \dots, X_N)$

The CDF of U is

$$F_U(u) = P\left[\max(X_1, \dots, X_N) \leq u\right]$$

$$= \sum_{n=1}^{\infty} P\left[\max(X_1, \dots, X_N) \leq u \,\middle|\, N = n\right] P(N=n)$$

$$= \sum_{n=1}^{\infty} \prod_{j=1}^{n} F_j(u) \; P(N=n) .$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} f_k(u) \prod_{\substack{j=1 \\ j \neq k}}^{n} F_j(u) \right] P(N=n)$$

The mth moment of U is

$$E[U^m] = \int_{-\infty}^{\infty} u^m f_U(u)\, du$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left[\int_{-\infty}^{\infty} u^m f_k(u) \prod_{\substack{j=1 \\ j \neq k}}^{n} F_j(u)\, du \right] P(N=n) .$$

3) $V = \min(X_1, X_2, \ldots, X_N)$

The CDF of V is

$$F_V(v) = P[\min(X_1, \ldots, X_N) \leq v]$$

$$= 1 - P[\min(X_1, \ldots, X_N) > v]$$

$$= 1 - \sum_{n=1}^{\infty} P[\min(X_1, \ldots, X_N) > v / N = n] \, P(N=n)$$

$$= 1 - \sum_{n=1}^{\infty} \prod_{j=1}^{n} [1 - F_j(v)] \, P(N=n).$$

The PDF of V is

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_k(v) \prod_{\substack{j=1 \\ j \neq k}}^{n} [1 - F_j(v)] \, P(N=n)$$

The m th moment of V is

$$E[V^m] = \int_{-\infty}^{\infty} v^m f_V(v) \, dv$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left[\int_{-\infty}^{\infty} v^m f_k(v) \prod_{\substack{j=1 \\ j \neq k}}^{n} [1 - F_j(v)] \, dv \right] P(N=n).$$

Ex 1 Suppose $X_i \sim Exp(a_i)$, $i = 1, 2, \ldots, n$

Find the distribution of U and V.

The CDF of U is

$$F_U(u) = \prod_{j=1}^{n} F_j(u)$$

$$= \prod_{j=1}^{n} \left[1 - e^{-a_j u}\right]$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{k=1}^{n} a_k e^{-a_k u} \prod_{\substack{j=1 \\ j \neq k}}^{n} \left[1 - e^{-a_j u}\right]$$

The CDF of V is

$$F_V(v) = 1 - \prod_{j=1}^{n} \left[1 - F_j(v)\right]$$

$$= 1 - \prod_{j=1}^{n} e^{-a_j v}$$

$$= 1 - e^{-(a_1 + a_2 + \cdots + a_n) v}$$

The PDF of V is

$$f_V(v) = (a_1 + \cdots + a_n) e^{-(a_1 + \cdots + a_n) v}$$

$\Rightarrow$ $V \sim Exp(a_1 + \cdots + a_n)$.

Mean of $V = \dfrac{1}{a_1 + \cdots + a_n}$

Variance of $V = \dfrac{1}{(a_1 + \cdots + a_n)^2}$

Ex 2

Suppose X_i are IID $Exp(a)$ for $i = 1, 2, \dots, N$.

$N \sim$ Trucated Poisson (λ) independent of $X_1, X_2, \dots$

Poisson: $P(X=k) = \dfrac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, \dots$

Truncated Poisson $P(X=k) = \dfrac{e^{-\lambda}\lambda^k}{(1-e^{-\lambda})\,k!}, \quad k = 1, 2, \dots$

Find the distributions of U and V

The CDF of U is

$$F_U(u) = \sum_{n=1}^{\infty}\left[\prod_{j=1}^{n} F_j(u)\right] P(N=n)$$

$$= \sum_{n=1}^{\infty} \left(1-e^{-au}\right)^n \frac{e^{-\lambda}\lambda^n}{(1-e^{-\lambda})\,n!}$$

$$= \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{n=1}^{\infty} \frac{\left[\left(1-e^{-au}\right)\lambda\right]^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$$

$$= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left\{ e^{(1-e^{-au})\lambda} - 1 \right\}$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \frac{1}{1-e^{-\lambda}} e^{-\lambda e^{-au}} \lambda a e^{-au}$$

$$= \frac{\lambda a}{1-e^{-\lambda}} e^{-\lambda e^{-au} - au}$$

The CDF of V is

$$F_V(v) = 1 - \sum_{n=1}^{\infty} \prod_{j=1}^{n} [1 - F_j(v)] \quad P(N=n)$$

$$= 1 - \sum_{n=1}^{\infty} e^{-nav} \frac{e^{-\lambda} \lambda^n}{(1-e^{-\lambda}) n!}$$

$$= 1 - \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{n=1}^{\infty} \frac{(e^{-av}\lambda)^n}{n!}$$

$$= 1 - \frac{1}{e^{\lambda}-1} \left[e^{e^{-av}\lambda} - 1 \right]$$

The PDF of V is

$$f_V(v) = \frac{d}{dv} F_V(v)$$

$$= \frac{\lambda a}{(e^{\lambda}) - 1} e^{-av} e^{\lambda e^{-av}}$$

After reading week...

Financial Risk Measures

1) Value at Risk
2) Expected Shortfall.