LECTURE

23 OCTOBER

9:00-10:00AM

MATH3/4/68181

Case vi $X_1, X_2, \ldots, X_N$ are dependent RVs,

N is a RV indep of $X_1, X_2, \ldots, X_N$

1) $S = X_1 + X_2 + \cdots + X_N$

The CDF of S is

$$F_S(s) = P(X_1 + \cdots + X_N \leq s)$$

$$= \sum_{n=1}^{\infty} P(X_1 + \cdots + X_N \leq s \mid N=n)\, P(N=n)$$

Total Probability Rule

$$= \sum_{n=1}^{\infty} \left[\int \cdots \int_{x_1 + \cdots + x_n \leq s} f(x_1, \ldots, x_n)\, dx_n \cdots dx_1 \right] P(N=n)$$

The PDF of S is

$$f_S(s) = \sum_{n=1}^{\infty} \left[\int \cdots \int_{x_1 + \cdots + x_n = s} f(x_1, \ldots, x_n)\, dx_n \cdots dx_1 \right] P(N=n)$$

The mth moment of S is

$$E[S^m] = \sum_{n=1}^{\infty} E[S^m \mid N=n]\, P(N=n)$$

$$= \sum_{n=1}^{\infty} \sum_{p_1 + \cdots + p_n = m} \binom{m}{p_1\, p_2 \cdots p_n} E[X_1^{p_1} \cdots X_n^{p_n}]\, P(N=n)$$

2) $U = \max(X_1, X_2, \ldots, X_N)$

The CDF of U is

$$F_U(u) = P[\max(X_1, \ldots, X_N) \leqslant u]$$

$$= \sum_{n=1}^{\infty} P[\max(X_1, \ldots, X_N) \leqslant u \mid N=n]\, P(N=n)$$

$$= \sum_{n=1}^{\infty} F\underbrace{(u, \ldots, u)}_{u \text{ repeated } n \text{ times}} P(N=n)$$

The PDF of U is

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \sum_{n=1}^{\infty} \left[\frac{d}{du} F(u, \ldots, u)\right] P(N=n)$$

The mth moment of U is

$$E(U^m) = \int_{-\infty}^{\infty} u^m f_U(u)\, du$$

$$= \sum_{n=1}^{\infty} \left[\int_{-\infty}^{\infty} u^m \frac{d}{du} F(u, \ldots, u)\right] P(N=n)$$

3) $V = \min(X_1, \dots, X_N)$

The CDF of V is

$$F_V(v) = P\left[\min(X_1, \dots, X_N) \leq v\right]$$

$$= \sum_{n=1}^{\infty} P\left[\min(X_1, \dots, X_N) \leq u \,\middle|\, N = n\right] P(N = n)$$

$$= \sum_{n=1}^{\infty} \left[1 - \bar{F}(\underbrace{v, \dots, v}_{v \text{ repeated } n \text{ times}})\right] P(N = n)$$

$$= \sum_{n=1}^{\infty} P(N = n) - \sum_{n=1}^{\infty} \bar{F}(v, \dots, v) P(N = n)$$

$$= 1 - \sum_{n=1}^{\infty} \bar{F}(v, \dots, v) P(N = n)$$

The PDF of V is

$$f_V(v) = -\sum_{n=1}^{\infty} \left[\frac{d}{dv} \bar{F}(v, \dots, v)\right] P(N = n)$$

The m th moment of V is

$$E[V^m] = -\sum_{n=1}^{\infty} \left\{\int_{-\infty}^{\infty} v^m \left[\frac{d}{dv} \bar{F}(v, \dots, v)\right] dv\right\} P(N = n)$$

Ex 1 Suppose a portfolio has 2 investments. Suppose

$$\bar{F}(x_1, x_2) = e^{-x_1 - x_2 - \theta x_1 x_2},$$

$$x_1 > 0$$
$$x_2 > 0$$

Find the distributions of

a) S
b) U
c) V

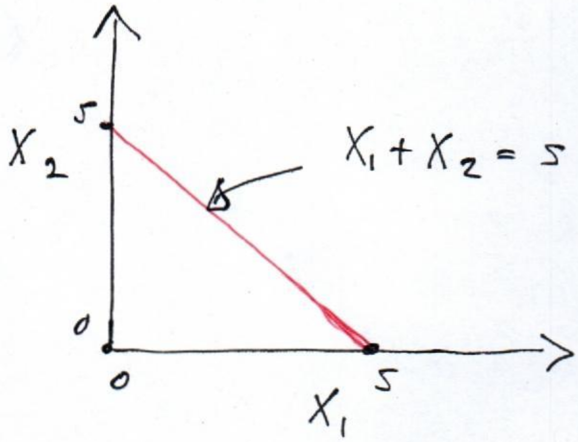
The joint PDF of (X_1, X_2) is

$$f(x_1, x_2) = \frac{(-1)^2 \, \partial^2}{\partial x_1 \partial x_2} e^{-x_1 - x_2 - \theta x_1 x_2}$$

$$= \frac{\partial}{\partial x_1} \left[(-1 - \theta x_1) e^{-x_1 - x_2 - \theta x_1 x_2} \right]$$

$$= (-1 - \theta x_1)(-1 - \theta x_2) e^{-x_1 - x_2 - \theta x_1 x_2}$$
$$- \theta e^{-x_1 - x_2 - \theta x_1 x_2}$$

$$= \left[1 - \theta + \theta(x_1 + x_2) + \theta^2 x_1 x_2 \right] e^{-x_1 - x_2 - \theta x_1 x_2}$$



a) $$f_S(s) = \int_0^s f(x_1, s - x_1)\, dx_1$$

$$= \int_0^s \left[1 - \theta + \theta s + \theta^2 x_1 (s - x_1)\right] e^{-s - \theta x_1 (s - x_1)}\, dx_1$$

$$= e^{-s}(1 - \theta + \theta s) \int_0^s e^{-\theta x_1 (s - x_1)}\, dx_1$$

$$+\, e^{-s}\theta^2 \int_0^s x_1 (s - x_1)\, e^{-\theta x_1 (s - x_1)}\, dx_1$$

<u>Homework</u> : Try working out these 2 integrals

b) The CDF of U is

$$F_U(u) = F(u, u)$$

$$= 1 - \bar{F}(u, -\infty) - \bar{F}(-\infty, u) + \bar{F}(u, u)$$

$$= 1 - e^{-u} - e^{-u} + e^{-2u - \theta u^2}$$

The PDF of U is

$$f_U(u) = 2e^{-u} - (2 - 2\theta u)e^{-2u - \theta u^2}$$

c) The CDF of V is

$$F_V(v) = 1 - \bar{F}(v, v)$$

$$= 1 - e^{-2v - \theta v^2}$$

The PDF of V is

$$f_V(v) = (2 + 2\theta v)e^{-2v - \theta v^2}$$

Ex 2 Suppose a portfolio has 3 investment. Suppose also

$$F(x_1, x_2, x_3) = \left[1 + x_1 + x_2 + x_3\right]^{-a},$$

$$a > 0$$
$$x_1 > 0$$
$$x_2 > 0$$
$$x_3 > 0$$

Find the distributions of

a) S

b) U

c) V

Homework